2nd March:

Def: $(\Omega, \mathcal{F}, \mu)$ is a measure space, if $f: \Omega \to \bar{\mathbb{R}}$ is measurable and $A \in \mathcal{F}$ then

$$\int_A f \, d\mu = \int_\Omega f \cdot \chi_A \, d\mu$$

χ_A = indicator/characteristic function

Def: $(\Omega, \mathcal{F}, \mu)$ is a measure space, suppose $f: \Omega \to \bar{\mathbb{R}}$ is measurable, f is said to satisfy property P almost everywhere (a.e) wrt μ if $\exists E \in \mathcal{F}$ with $\mu(E) = 0$ s.t f satisfies P for all $\omega \in E^c = \Omega \setminus E$

f can also be true on E

Lemma: Suppose $f, g: \Omega \to \bar{\mathbb{R}}$ is such that $f = g$ (a.e). If $(\Omega, \mathcal{F}, \mu)$ is a complete measure space, then f is measurable iff g is measurable

proof:

($\Rightarrow$) f is measurable, then $\exists E \in \mathcal{F}$ s.t $\mu(E) = 0$ and $f = g$ on E^c i.e

$$E^c \subseteq \{\omega \in \Omega \mid f(\omega) = g(\omega)\}$$

consider $A \in \mathcal{B}(\bar{\mathbb{R}})$, as f is measurable, we have $f^{-1}(A) \in \mathcal{F}$

now $g^{-1}(A) = \{\omega \in \Omega \mid g(\omega) \in A\}$

$= (\{\omega \in \Omega \mid g(\omega) \in A\} \cap E^c) \cup (\{\omega \in \Omega \mid g(\omega) \in A\} \cap E)$

$\{\omega \in \Omega \mid f(\omega) \in A\} \cap E^c$

$\subseteq E$ and $\mu(E) = 0$ $\Rightarrow \{\omega \in \Omega \mid g(\omega) \in A\} \cap E \in \mathcal{F}$ as complete

$\Rightarrow g^{-1}(A) \in \mathcal{F}$ as it is union of two measurable sets

($\Leftarrow$) same as above, by interchanging g with f and f with g

Eg: suppose $E \in \mathcal{B}(\mathbb{R})$ s.t $\mu(E) = 0$ and $F \subseteq E$ with $F \notin \mathcal{B}(\mathbb{R})$, $F \in \mathcal{L}$ as it is complete

$\Rightarrow F \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$

Consider $\chi_F: \mathbb{R} \to \mathbb{R}$ then $\chi_F = 0$ a.e as

$\exists F$ s.t $\mu(F) = 0$, $F \in \mathcal{L}$ now

$\chi_F(\omega) = 0 \quad \forall \omega \in F^c$ but

χ_F is not Borel measurable function as

$\chi_F^{-1}(\mathbb{R}) = F \notin \mathcal{B}(\mathbb{R})$

Note: above example shows that $f = g \Rightarrow f$ is measurable, even though g is measurable (as not complete measure space)

Theorem: Suppose $(\Omega, \mathcal{F}, \omega)$ is a measure space, let $A, B \in \mathcal{F}$, $A \cap B = \phi$, If $f, g: \Omega \to \bar{\mathbb{R}}$ are integrable, then f is also integrable over A, $f + g$ now well defined and $|f|$ are also integrable and

(1) $\int_\Omega (f+g) \, d\mu = \int_\Omega f \, d\mu + \int_\Omega g \, d\mu$

(2) $\int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu$

(3) f is finite a.e

(4) If $c \in \mathbb{R}$ (not $\bar{\mathbb{R}}$) then cf is integrable and

$$\int_\Omega cf\,d\mu = c\int_\Omega f\,d\mu$$

(5) If $f \geq g$ (a.e) then

$$\int_\Omega f\,d\mu \geq \int_\Omega g\,d\mu$$

(6) $\left|\int_\Omega f\,d\mu\right| \leq \int_\Omega |f|\,d\mu$

(7) If $f \geq 0$ with $\int_\Omega f\,d\mu = 0$ then $f = 0$ (a.e)

(8) If $f = g$ (a.e) then

$$\int_\Omega f\,d\mu = \int_\Omega g\,d\mu$$

(9) If $h: \Omega \to \bar{\mathbb{R}}$ is measurable and $|h(\omega)| \leq f(\omega)$ $\forall \omega \in \Omega$, then h is also integrable

proof: It will follow from tutorial

Note: The set of integrable functions form some sort of vector space over $\mathbb{R}$

Theorem: (Lebesgue's dominated convergence theorem) Suppose $f_n: \Omega \to \bar{\mathbb{R}}$ is sequence of measurable functions s.t

$$f(\omega) = \lim_{n\to\infty} f_n(\omega) \quad \forall \omega \in \Omega \text{ (a.e also works here)}$$

then if $\exists\, g: \Omega \to [0,\infty]$ s.t $|f_n(\omega)| \leq g(\omega)$ $\forall n \in \mathbb{N}$, $\forall \omega \in \Omega$ (a.e also works here) and g is integrable, then f is integrable and

$$\lim_{n\to\infty} \int_\Omega |f_n - f|\,d\mu = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_\Omega f_n\,d\mu = \int_\Omega f\,d\mu$$

proof: as g is integrable then f_n, f also integrable by → part (9) of above theorem

and as $|f_n| \leq g$, $|f| \leq g$

$$\Rightarrow |f_n - f| \leq 2g$$

consider $\{2g \pm |f_n - f|\}_{n=1}^{\infty}$ and apply fatou's lemma

$$\Rightarrow \int_\Omega 2g\,d\mu = \int_\Omega \liminf_{n\to\infty} (2g \pm |f_n - f|)\,d\mu$$

$$= \int_\Omega \liminf_{n\to\infty} (2g \pm |f_n - f|)\,d\mu$$

$$\leq \liminf_{n\to\infty} \int_\Omega (2g \pm |f_n - f|)\,d\mu \quad (\because \text{fatou's lemma})$$

$$\leq \int_\Omega 2g\,d\mu + \liminf_{n\to\infty} \left(\pm \int_\Omega |f_n - f|\,d\mu\right)$$

as g is integrable, so that integral as constant

$$\Rightarrow 0 \leq \lim_{n\to\infty} \left(\pm \int_\Omega |f_n - f| \, d\mu \right)$$

$$\Rightarrow \lim_{n\to\infty} \left(\int_\Omega |f_n - f| d\mu \right) \geq 0 \quad \text{——} (1)$$

$$\& \lim_{n\to\infty} \left(- \int_\Omega |f_n - f| d\mu \right) \geq 0$$

$$\Rightarrow \overline{\lim_{n\to\infty}} \left(\int_\Omega |f_n - f| d\mu \right) \leq 0 \quad \text{——} (2)$$

from (1), (2) $\lim_{n\to\infty} \int_\Omega |f_n - f| d\mu \geq 0 \geq \overline{\lim_{n\to\infty}} \int_\Omega |f_n - f| d\mu$

$$\Rightarrow \lim_{n\to\infty} \int_\Omega |f_n - f| d\mu = 0$$

$$\Rightarrow \left| \int_\Omega f_n d\mu - \int_\Omega f d\mu \right| \leq \int_\Omega |f_n - f| d\mu = 0 \text{ as } n \to \infty$$

$$\Rightarrow \lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu$$

ex: suppose $(\Omega, \mathcal{F}, \mu)$ is a σ-finite measure space, and $f: \Omega \to [0, \infty)$ is a measurable function, show that

$\mu_f(E) = \int_E f d\mu \quad E \in \mathcal{F}$ → done down

is σ-finite measure on $(\Omega, \mathcal{F})$

Theorem: If $f: [a,b] \to \mathbb{R}$ is Riemann integrable, then $\int_a^b f(x)dx = \int f d\lambda$
so, every riemann integrable function is lebesgue integrable $[a,b]$

proof:

as f is riemann integrable on $[a,b]$, $\exists M > 0$ s.t

$$|f(x)| \leq M \quad \forall x \in [a,b]$$

for $n \in \mathbb{N}$ consider partition of $[a,b]$ follows:

$$P_n = \{a = x_0 < x_1 < \cdots < x_{k(n)} = b\}$$

s.t

$$\|P_n\| \to 0 \text{ as } n \to \infty$$

then $\int_a^b f(x)dx = \lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} U(P_n, f)$

where $L(P_n, f) = \sum_{i=1}^{k(n)} m_i^\circ (x_i - x_{i-1})$

$$m_i^\circ = \inf_{k(n)} \{f(x) \mid x_{i-1} \leq x \leq x_i^\circ\}$$

$$U(P_n, f) = \sum_{i=1}^{k(n)} M_i^\circ (x_i^\circ - x_{i-1}^\circ) \quad M_i^\circ = \sup \{f(x) \mid x_{i-1} \leq x \leq x_i\}$$

now, let $f_n = \sum_{i=1}^{k(n)} m_i^\circ \chi_{[x_{i-1}^\circ, x_i)}$ $\quad g_n = \sum_{i=1}^{k(n)} M_i^\circ \chi_{[x_{i-1}, x_i)}$

then $\int_{\mathbb{R}} f_n d\lambda = \int_{[a,b]} f_n \, d\lambda = L(P_n, f)$ ($\because f_n, g_n$ are basic)

$$\int_{\mathbb{R}} g_n \, d\lambda = \int_{[a,b]} g_n \, d\lambda = U(P_n, f)$$

also $f_n(x) \le f(x) \le g_n(x) \quad \forall x \in [a,b]$

also, $\int_{[a,b]} g_n \, d\lambda - \int_{[a,b]} f_n \, d\lambda = U(P_n, f) - L(P_n, f)$

↳ same as $\mathbb{R}$ as Ω everywhere else

$$\Rightarrow \lim_{n\to\infty} \int_{\mathbb{R}} (g_n - f_n) \, d\lambda = 0$$

$$\Rightarrow \int_{\mathbb{R}} \lim_{n\to\infty} (g_n - f_n) \, d\lambda = 0 \quad (\because \text{fatou lemma as } g_n \ge f_n)$$

$$\Rightarrow \lim_{n\to\infty} g_n = \overline{\lim_{n\to\infty}} f_n \quad (a.e)$$

$$\Rightarrow \lim_{n\to\infty} f_n = f \quad (a.e) \quad \text{for } x \in [a,b]$$

also $\exists M$ s.t $|f_n| \le M$

$$\Rightarrow \lim_{n\to\infty} \int_{[a,b]} f_n \, d\lambda = \int_{[a,b]} f \, d\lambda$$

$$\Rightarrow \int_a^b f \, dx = \int_{[a,b]} f \, d\lambda \quad (\because \text{DCT, by restricting to } \Omega = [a,b])$$

ex: suppose $(\Omega, \mathcal{H}, \mu)$ is a σ-finite measure space, and $f: \Omega \to [0,\infty)$ is a measurable function, show that

$$\mu_f(E) = \int_E f \, d\mu \quad E \in \mathcal{H}$$

is σ-finite measure on $(\Omega, \mathcal{H})$

proof:

$$\mu_f(\phi) = \int_\Omega f \chi_\phi \, d\mu$$

$$= \int_\Omega 0 \, d\mu \quad (\because \chi_\phi \equiv 0)$$

$$= 0 \quad \text{---} (1)$$

also $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ s.t $i \ne j \Rightarrow E_i^\circ \cap E_j^\circ = \phi$

$$E = \bigcup_{i=1}^{\infty} E_i^\circ$$

also, $\chi_E = \chi_{E_1} + \chi_{E_2} + \cdots$ as all disjoint

$$\Rightarrow \int_\Omega f \chi_E \, d\mu = \int_\Omega f(\chi_{E_1} + \chi_{E_2} + \cdots) \, d\mu$$

$$= \int_\Omega f\chi_{E_1} \, d\mu + \int_\Omega f\chi_{E_2} \, d\mu + \cdots$$

$$\Rightarrow \mu_f(E) = \sum_{i=1}^{\infty} \mu_f(E_i^\circ) \quad \text{---} (2)$$

also, $\exists \{E_i\}_{i=1}^{\infty}$ s.t $\mu(E_i) < \infty \ \forall i \in \mathbb{N}$

$$\Omega = \bigcup_{i=1}^{\infty} E_i$$

then $\mu_f(E_i) = \int_{\Omega} f \chi_{E_i} d\mu \quad \mu(E_i) < \infty$

as f is measurable $\exists \{f_n\}_{n(m)}$ basic functions

then $\int_{\Omega} f_n \chi_{E_i} d\mu = \sum_{j=1}^{\infty} c_j \mu(F_j \cap E_i) < \infty$

$c_j \in \mathbb{R} > 0$ as $\mu(E_i) < \infty$

$\Rightarrow \lim_{n \to \infty} \int_{\Omega} f_n \chi_{E_i} d\mu < \infty$

$\Rightarrow \int_{\Omega} f \chi_{E_i} d\mu < \infty$

$\Rightarrow \mu_f(E_i) < \infty \ \forall i \in \mathbb{N}$

so, $\exists \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ s.t $\Omega = \bigcup_{i=1}^{\infty} E_i$ — ③

and $\mu_f(E_i) < \infty$

from ①, ②, ③ μ_f is a σ-finite measure on $(\Omega, \mathcal{A})$

9th March:

Extra class wednesday 5pm to 7pm, MA 114

product measure:

For $i=1,2$

$(\Omega_i, \mathcal{H}_i, \mu_i)$ is a (σ-finite) measure space ← we assume this in case of product measure

$\Omega = \Omega_1 \times \Omega_2$

the problem is product of σ-algebra is not a σ-algebra

Lemma: $\mathcal{H}_1 \times \mathcal{H}_2$ is a semi-algebra on Ω

proof:

By definition, $\Omega = \Omega_1 \times \Omega_2 \in \mathcal{H}_1 \times \mathcal{H}_2$ —— ①

suppose $E_1 \times F_1,\ E_2 \times F_2 \in \mathcal{H}_1 \times \mathcal{H}_2$

then, $(E_1 \times F_1) \cap (E_2 \times F_2)$

$= (E_1 \cap E_2) \times (F_1 \cap F_2)$

$\in \mathcal{H}_1 \times \mathcal{H}_2$ —— ②

and, $(E_1 \times F_1)^c = \Omega \setminus (E_1 \times F_1)$

$= (\Omega_1 \setminus E_1 \times F_1) \cup (E_1 \times \Omega_2 \setminus F_1) \cup (\Omega_1 \setminus E_1 \times \Omega_2 \setminus F_2)$

each of them are in $\mathcal{H}_1 \times \mathcal{H}_2$ and pairwise disjoint —— ③

by ①, ②, ③ $\mathcal{H}_1 \times \mathcal{H}_2$ is a semi-algebra

Defn: The product σ-algebra on Ω is the σ-algebra generated by $\mathcal{H}_1 \times \mathcal{H}_2$ it is denoted by $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

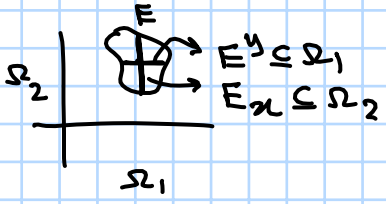
the elements of $\mathcal{H}_1 \times \mathcal{H}_2$ are called measurable rectangles

Defn: If $E \subseteq \Omega_1 \times \Omega_2$ then for each $x \in \Omega_1$, $y \in \Omega_2$ we define

$E_x = \{\tilde{y} \in \Omega_2 \mid (x, \tilde{y}) \in E\}$

$E^y = \{\tilde{x} \in \Omega_1 \mid (\tilde{x}, y) \in E\}$

E_x is called x-section, E^y is called y-section



Defn: If $f: \Omega \to \bar{\mathbb{R}}$, then for each $x \in \Omega_1$ and $y \in \Omega_2$ we define x-section of f as

$f_x: \Omega_2 \to \bar{\mathbb{R}}$ by $f_x(\tilde{y}) = f(x, \tilde{y})$

and y-section of f as: $f^y: \Omega_1 \to \bar{\mathbb{R}}$ by $f^y(\tilde{x}) = f(\tilde{x}, y)$

Suppose $E = A \times B \subseteq \Omega_1 \times \Omega_2$

$(x,y) \in \Omega_1 \times \Omega_2$

if $A = \phi$, $x \in \Omega_1 \neq \phi$ $E_x = \{ \tilde{y} \in \Omega_2 \mid (x, \tilde{y}) \in A \times B \}$

$= \{ \tilde{y} \in \Omega_2 \mid (x, \tilde{y}) \in \phi \times B \}$

$= \phi$ as no $x \in \phi$

so, $E_x = \begin{cases} B ; & x \in A \\ \phi ; & x \notin A \end{cases}$ $\quad E_y = \begin{cases} A ; & y \in B \\ \phi ; & y \notin B \end{cases}$

so, given a rectangle, the E_x will give us second component if $x \in A$ or first component

Lemma: If $E \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ then $\forall x \in \Omega_1$, $\forall y \in \Omega_2$ $E_x \in \mathcal{H}_2$ and $E^y \in \mathcal{H}_1$

proof: let $E \in \mathcal{H}$ then $(x,y) \in \Omega_1 \times \Omega_2$

$$\mathcal{C} = \left\{ E \in \mathcal{H} \;\middle|\; E_x \in \mathcal{H}_2 \ \forall x \in \Omega_1,\ E^y \in \mathcal{H}_1, \forall y \in \Omega_2 \right\}$$

if we show $\mathcal{C}$ is a σ-algebra

then as $\mathcal{H}_1 \times \mathcal{H}_2 \subseteq \mathcal{C} \subseteq \mathcal{H}$

$\Rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2 \subseteq \mathcal{H}(\mathcal{C}) \subseteq \mathcal{H}$

$\Rightarrow \mathcal{H} = \mathcal{C}$

now, $\mathcal{C}$ is a σ-algebra as

① $\Omega \in \mathcal{C}$ as $\Omega = \Omega_1 \times \Omega_2$ and

$\Omega_x = \{ \tilde{y} \in \Omega_2 \mid (x, \tilde{y}) \in \Omega_1 \times \Omega_2 \}$

$= \Omega_2 \in \mathcal{H}_2$

$\Omega^y = \Omega_1 \in \mathcal{H}_1$

② for $x \in \Omega_1$, $E \in \mathcal{C}$

$(E^c)_x = \{ (x, \tilde{y}) \in \Omega_2 \mid (x, \tilde{y}) \in E^c \}$

$z \in (E^c)_x \Leftrightarrow (x,z) \in E^c \Leftrightarrow (x,z) \notin E \Leftrightarrow z \notin E_x \Leftrightarrow z \in (E_x)^c$

so, $(E^c)_x = (E_x)^c$ similarly

$(E^c)^y = (E^y)^c$

and so, $E^c \in \mathcal{C}$ as $(E^c)_x \in \mathcal{H}_1$, $(E^c)^y \in \mathcal{H}_2$

③ let $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{C}$, for $x \in \Omega_1$

$\left(\bigcup_{i=1}^{\infty} E_i\right)_x = \bigcup_{i=1}^{\infty} (E_i^y)_x \in \mathcal{H}_2$

as $y \in \left(\bigcup_{i=1}^{\infty} E_i\right)_x \Leftrightarrow (x,y) \in \bigcup_{i=1}^{\infty} E_i \Leftrightarrow \exists i, (x,y) \in E_i \Leftrightarrow \exists i, y \in (E_i)_x \Leftrightarrow y \in \bigcup_{i=1}^{\infty} (E_i)_x$

similarly for $y \in \Omega_2$ $\left(\bigcup_{j=1}^{\infty} E_j\right)^y = \bigcup_{j=1}^{\infty} (E_j)^y \in \mathcal{H}_2$

so, $\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{C}_c$

Note: Converse of above lemma is not true

Theorem: If $\mu : \mathcal{H}_1 \times \mathcal{H}_2 \to [0, \infty]$ is defined by $\mu(E \times F) = \mu_1(E)\mu_2(F)$, then μ is σ-additive and σ-finite

proof: Suppose $A \in \mathcal{H}_1 \times \mathcal{H}_2$ and

$\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{H}_1 \times \mathcal{H}_2$ is a disjoint collection s.t

$$A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{H}_1 \times \mathcal{H}_2$$

let $A = E \times F$, $A_i = E_i \times F_i \ \forall\, i \in \mathbb{N}$

so, $E_i, E \in \mathcal{H}_1 \quad F_i, F \in \mathcal{H}_2 \quad \forall i \in \mathbb{N}$

observe that $E = \bigcup_{i=1}^{\infty} E_i \quad F = \bigcup_{i=1}^{\infty} F_i$ as an disjoint — ①

and for $j \neq k$

$$(A_j)_x \cap (A_k)_x = \phi \quad — ②$$

now, $\mu(A) = \sum_{i=1}^{\infty} \mu(A_i)$ then we are done

$$\mu(A) = \mu(E)\,\mu(F)$$

$$\mu(A_i) = \mu(E_i)\,\mu(F_i)$$

For $i \in \mathbb{N}$, $A_x = \begin{cases} F : x \in E \\ \phi ; x \notin E \end{cases}$

$$(A_i)_x = \begin{cases} F_i ; x \in E_i \\ \phi ; x \notin E_i \end{cases}$$

so, if $x \in E$, then $F = A_x = \left(\bigcup_{i=1}^{\infty} A_i\right)_x$

$$= \bigcup_{i=1}^{\infty} (A_i)_x$$

so, $F = \bigcup_{i=1}^{\infty} (A_i)_x = \bigcup_{\{j \mid x \in E_j\}} F_j$, when $x \in E$

is a disjoint union as if $\exists\, y \in F_j \cap F_k$, then $(x, y) \in (E_j \times F_j) \cap (E_k \times F_k)$
($\because$ ②)

$(x, y) \in \phi$ this is a contradiction — $E_j \times F_j$ disjoint

so, $\mu_2(F) = \sum_{\{j \mid x \in E_j\}} \mu_2(F_j)$ ($\because \mu_2$ is a measure)

if $x \notin E$ then $A_x = \phi$

$$\Rightarrow (A_i)_x = \phi \quad \forall\, i \in \mathbb{N}$$

so, $\mu_2(F)\chi_E(x) = \sum_{j=1}^{\infty} \mu_2(F_j)\chi_{E_j}(x) \quad \forall x \in \Omega_1$ ($\because$ combining both cases)

integrate both sides wrt μ_1:

$$\int_{\Omega_1} \mu_2(F)\chi_E(x)\,d\mu_1(x) = \int_{\Omega_1} \sum_{j=1}^{\infty} \mu_2(F_j)\chi_{E_j}(x)\,d\mu_2(x)$$

$$\Rightarrow \mu_2(F)\mu_1(E) = \sum_{j=1}^{\infty} \int_{\Omega_1} \mu_2(F_j)\chi_{E_j}(x)\,d\mu_1(x)$$

$$= \sum_{j=1}^{\infty} \mu_2(F_j)\mu_1(E_j)$$

$$\Rightarrow \mu(A) = \sum_{j=1}^{\infty} \mu(A_j) \text{ and so } \mu \text{ is a measure}$$

11th March:

Theorem: If $\mu: \mathcal{H}_1 \times \mathcal{H}_2 \to [0,\infty]$ is defined by $\mu(E\times F)=\mu_1(E)\mu_2(F)$, then μ is σ-additive and σ-finite

proof:

we have already seen μ is a σ-additive

i.e $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{H}_1 \times \mathcal{H}_2$ $A_i \cap A_k = \phi$ $i \neq k$

then $\sum_{i=1}^{\infty} \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$

as $(\Omega_i, \mathcal{H}_i, \mu_i)$ is σ-finite for $i=1,2$

then $\exists \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{H}_1$ & $\{F_j\}_{j=1}^{\infty} \subseteq \mathcal{H}_2$ s.t

$\mu_1(E_i)<\infty$, $E_i \subseteq E_{i+1}$ $\forall i \in \mathbb{N}$ and $\Omega_1 = \bigcup_{i=1}^{\infty} E_i$

$\mu_2(F_j)<\infty$, $F_j \subseteq F_{j+1}$ $\forall j \in \mathbb{N}$ and $\Omega_2 = \bigcup_{j=1}^{\infty} F_j$

then, $\Omega = \bigcup_{k=1}^{\infty} (E_k \times F_k)$ with $\mu(E_k \times F_k) = \mu_1(E_k)\mu_2(F_k) < \infty$ $\forall k \in \mathbb{N}$

$E_k \times F_k \subseteq E_{k+1} \times F_{k+1}$ $\forall k \in \mathbb{N}$ (so we don't have to reindex)

$\Rightarrow \mu$ is σ-finite

Note: As $\mathcal{H}_1 \times \mathcal{H}_2$ is sigma algebra, corollary extension of

$\mu: \mathcal{H}_1 \times \mathcal{H}_2 \to [0,\infty]$

$\mu_1 \otimes \mu_2: \mathcal{H}_1 \otimes \mathcal{H}_2 \to [0,\infty]$ is unique

this $\mu_1 \otimes \mu_2: \mathcal{H}_1 \otimes \mathcal{H}_2 \xrightarrow{(\because \mu \text{ is } \sigma\text{-finite})} [0,\infty]$ is called the product measure

Note: In order to define measurable space, we don't need any μ as just $\mathcal{H}_1 \otimes \mathcal{H}_2$

Exe: Show that $\mathcal{H}(S) = \mathcal{B}(\mathbb{R}^2)$ → done down

consider the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$, then $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$

as, $S = \{(a,b)\times(c,d) \mid a,b,c,d \in \mathbb{R}\}$

↑ collection of open rectangles

then $\mathcal{H}(S) = \mathcal{B}(\mathbb{R}^2)$ ($\because$ above exercise)

now, $S \subseteq \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$

$\Rightarrow \mathcal{H}(S) \subseteq \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$

$\Rightarrow \mathcal{B}(\mathbb{R}^2) \subseteq \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$

now, consider $\mathcal{C}_1 = \{A \subseteq \mathbb{R} \mid A \times \mathbb{R} \in \mathcal{B}(\mathbb{R}^2)\}$

$\mathcal{C}_2 = \{B \subseteq \mathbb{R} \mid \mathbb{R} \times B \in \mathcal{B}(\mathbb{R}^2)\}$

$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{C}_1$ as $\mathcal{C}_1$ contains all open subsets of $\mathbb{R}$ and it is a σ-algebra (trivial to see)

and so,

$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{C}_2$ by similar steps, then we get

$$\{A \times \mathbb{R} \mid A \in \mathcal{B}(\mathbb{R})\} \subseteq \mathcal{B}(\mathbb{R}^2)$$

$$\& \{\mathbb{R} \times B \mid B \in \mathcal{B}(\mathbb{R})\} \subseteq \mathcal{B}(\mathbb{R}^2)$$

$$\Rightarrow \{A \times B \mid A \in \mathcal{B}(\mathbb{R}), B \in \mathcal{B}(\mathbb{R})\} \subseteq \mathcal{B}(\mathbb{R}^2)$$

$$\Rightarrow \mathcal{H}(\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})) \subseteq \mathcal{B}(\mathbb{R}^2)$$

$$\Rightarrow \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}(\mathbb{R}^2)$$

so, $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$

Note: $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^m) = \mathcal{B}(\mathbb{R}^{n+m}) \quad \forall n, m \in \mathbb{N}$

now, $\mathcal{L}(\mathbb{R}^2) \supset \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$ i.e $\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$ not complete

as $A_1 \cup N_1$, $A_2 \cup N_2$ s.t (not always in $\mathcal{L}(\mathbb{R}^2)$)

$$(A_1 \cup N_1) \times (A_2 \cup N_2) = (A_1 \times A_2) \cup (A_1 \times N_2) \cup (A_2 \times N_1) \cup (N_1 \times N_2)$$

now, fix $(\Omega_i, \mathcal{H}_i, \mu_i)$, σ-finite measure spaces, and consider the product space $(\Omega, \mathcal{H}, \mu)$ the product measure space

$$\Omega = \Omega_1 \times \Omega_2$$
$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$
$$\mu = \mu_1 \otimes \mu_2 \text{ (the product measure)}$$

Lemma: If $f : \Omega \to \bar{\mathbb{R}}$ is a measurable function, then the functions $f_x : \Omega_2 \to \bar{\mathbb{R}}$ and $f^y : \Omega_1 \to \bar{\mathbb{R}}$ are $\mathcal{H}_2$-measurable and $\mathcal{H}_1$ measurable $\forall x \in \Omega_1, \forall y \in \Omega_2$

proof: Fix $B \in \mathcal{B}(\bar{\mathbb{R}})$ and $x \in \Omega_1$ then

$$f_x^{-1}(B) = \{y \in \Omega_2 \mid f(x,y) \in B\}$$

$$= \{y \in \Omega_2 \mid (x,y) \in f^{-1}(B)\}$$

$$\Rightarrow \{y \in \Omega_2 \mid (x,y) \in f^{-1}(B)\} \in \mathcal{H}_2 \quad (f^{-1}(B) \in \mathcal{H})$$

(∵ previously done theorem)

Theorem: Let $E \in \mathcal{H}$ and let $f : \Omega \to \bar{\mathbb{R}}$ be a function, then

1) the functions

$x \in \Omega_1 \longrightarrow \mu_2(E_x) \in \bar{\mathbb{R}}$ is $\mathcal{H}_1$-measurable

and $y \in \Omega_2 \longrightarrow \mu_1(E^y) \in \bar{\mathbb{R}}$ is $\mathcal{H}_2$-measurable

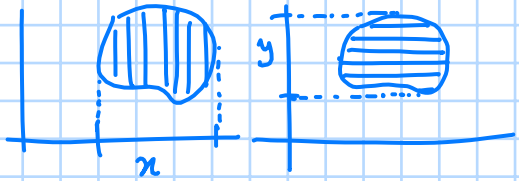
2) the following holds

$$\int_{\Omega_1} \mu_2(E_x)\, d\mu_1(x) = \mu(E) = \int_{\Omega_2} \mu_1(E^y)\, d\mu_2(y)$$

proof: **step 1**: the theorem is true for $E \in \mathcal{H}_1 \times \mathcal{H}_2$ as

let $E = A \times B$, $A \in \mathcal{H}_1$, $B \in \mathcal{H}_2$

then, $\mu_2(E_x) = \begin{cases} \mu_2(B) & ; x \in A \\ 0 & ; x \notin A \end{cases}$



$= \mu_2(B)\, \chi_A(x) \quad \forall x \in \Omega_1$

this works even if $\mu_2(B) = \infty$

$f : \Omega_1 \longrightarrow \bar{\mathbb{R}}$ s.t $f(x) = \mu_2(B)\, \chi_A(x)$

then as $\chi_A(x)$ is measurable

$\Rightarrow f(x) = \mu_2(B)\, \chi_A(x)$ is measurable

$$\Rightarrow \int_{\Omega_1} f(x)\, d\mu_1(x) = \int_{\Omega_1} \mu_2(B)\, \chi_A(x)\, d\mu_1(x)$$

$$= \mu_1(A)\, \mu_2(B)$$

$$= \mu(E)$$

step 2: the theorem is true for $E \in \mathcal{A}(\mathcal{H}_1 \times \mathcal{H}_2)$

$E = \bigcup_{i=1}^{n} E_i^\circ$, $E_i^\circ = A_i \times B_i$, $\{A_i\}_{i=1}^n \subseteq \mathcal{H}_1$, $\{B_i\}_{i=1}^n \subseteq \mathcal{H}_2$

and $E_k \cap E_l = \phi \ \forall k \neq l$ in $\{1, 2, \ldots n\}$

for $x \in \Omega_1$, $E_x = \left(\bigcup_{i=1}^{n} E_i^\circ\right)_x = \bigcup_{i=1}^{n} (E_i^\circ)_x$ ($\because$ seen previously)

and $(E_i^\circ)_x \cap (E_j^\circ)_x = \phi \ \forall \ i \neq j$ ($\because$ seen previously)

then, $\mu_2(E_x) = \sum_{i=1}^{n} \mu_2(B_i)\, \chi_{A_i}(x)$

so, $x \to \mu_2(E_x)$ is $\mathcal{H}_1$-measurable and

$$\int_{\Omega_1} \mu_2(E_x)\, d\mu_1(x) = \sum_{i=1}^{n} \mu_2(B_i)\, \mu_1(A_i) \quad \left(\because \int \sum_{i=1}^{n} = \sum_{i=1}^{n} \int\right)$$

$$= \sum_{i=1}^{n} \mu(E_i^\circ) = \mu(E)$$

Similarly for y

step 3: suppose $\mu_i^\circ(\Omega_i) < \infty$ for $i = 1, 2$

let $\mathcal{G} = \{E \in \mathcal{H} \mid (1) \text{ and } (2) \text{ holds for } E_x\}$ ← for good-set principle

if we know $\mathcal{G} \subseteq \mathcal{H}$ is σ-algebra

then we are done

as $\mathcal{H}_1 \times \mathcal{H}_2 \subseteq \mathcal{G}$ ($\because$ step 1)

$\Rightarrow \mathcal{H}(\mathcal{G}) = \mathcal{H} \Rightarrow \mathcal{G} = \mathcal{H}$

if we know $\mathcal{C}_\mathcal{G}$ is a monotone class, then from step 2

$$\mathcal{A}(\mathcal{F}_1 \times \mathcal{F}_2) \subseteq \mathcal{C}_\mathcal{G}$$

$\Rightarrow \mathcal{C}_\mathcal{G}$ is monotone and $\mathcal{A}(\mathcal{F}_1 \times \mathcal{F}_2) \subseteq \mathcal{C}_\mathcal{G}$

then $\mathcal{C}_\mathcal{G}$ is σ-algebra, so $\mathcal{C}_\mathcal{G} = \mathcal{F}_1$

now, $\{E^{(n)}\}_{n=1}^{\infty}$ is $\uparrow$ family in $\mathcal{C}_\mathcal{G}$, $E^{(n)} \uparrow E \in \mathcal{F}_1$

then, $(E^{(n)})_x \uparrow E_x \quad \forall x \in \Omega_1$

by continuity of μ_2 from below

$$\mu_2((E^{(n)})_x) \uparrow \mu_2(E_x) \quad \forall x \in \Omega_1$$

$\Rightarrow x \to \mu_2(E_x)$ is measurable as it is pointwise limit of measurable functions

so, by monotone-convergence theorem

$$\lim_{n\to\infty} \int_{\Omega_1} \mu_2(E_x^{(n)})\, d\mu_1(x) = \int_{\Omega_1} \mu_2(E_x)\, d\mu_1(x)$$

$$\Rightarrow \lim_{n\to\infty} \mu(E^{(n)}) = \int_{\Omega_1} \mu_2(E_x)\, d\mu_1(x)$$

by continuity from below for μ:

$$\mu(E) = \int_{\Omega_1} \mu_2(E_x)\, d\mu_1(x)$$

so, $E \in \mathcal{C}_\mathcal{G}$ ——— (1)

now $\{E^{(n)}\}_{n=1}^{\infty} \subseteq \mathcal{C}_\mathcal{G}$ s.t $E^{(n)} \downarrow E \in \mathcal{F}_1$

$\forall x \in \Omega_1 \quad (E^{(n)})_x \downarrow E_x$

as, $\mu_2(\Omega_2) < \infty$, using continuity

$$\mu_2((E^{(n)})_x) \downarrow \mu_2(E_x) \quad \forall x \in \Omega_1$$

as $\{E^{(n)}\}_{n=1}^{\infty} \subseteq \mathcal{C}_\mathcal{G}$, $x \to \mu_2(E_x)$ is $\mathcal{F}_1$-measurable (as integrals bounded)

by reverse monotone convergence/dominated conv theorem

$$\underbrace{x \to \mu_2(E_x^{(1)}) \leq \mu_2(\Omega_2)}_{\text{Bounded measurable function}}$$

so, $x \to \mu_2(E_x^{(1)})$ is integrable, by Dominated convergence theorem

$$\lim_{n\to\infty} \int_{\Omega_1} \mu_2(E_x^{(n)})\, d\mu_1(x) = \int_{\Omega_1} \mu_2(E_x)\, d\mu_1(x)$$

$$\Rightarrow \lim_{n\to\infty} \mu(E^{(n)}) = \int_{\Omega_1} \mu_2(E_x)\, d\mu_1(x)$$

as $\mu(\Omega) < \infty$, we use continuity from above for μ and

$$\Rightarrow \mu(E) = \int_{\Omega_1} \mu_2(E_x)\, d\mu_1(x)$$

$$\Rightarrow E \in \mathcal{C}_g \quad \text{---} \quad (2)$$

from (1), (2) $\mathcal{C}_g$ is monotone-class $\Rightarrow \mathcal{F}_1 = \mathcal{C}_g$

step 4: suppose $(\Omega_i, \mathcal{F}_i, \mu_i)$ is σ-finite for $i = 1, 2$

then $\exists$ increasing families

$$\{A^{(n)}\}_{n=1}^{\infty} \subseteq \mathcal{F}_1$$

$$\{B^{(n)}\}_{n=1}^{\infty} \subseteq \mathcal{F}_2$$

s.t $\Omega_1 = \bigcup_{n=1}^{\infty} A^{(n)}$ $\Omega_2 = \bigcup_{n=1}^{\infty} B^{(n)}$

$$\mu_1(A^{(n)}) < \infty, \ \mu_2(B^{(n)}) < \infty \ \forall n \in \mathbb{N}$$

observe that $\Omega = \bigcup_{n=1}^{\infty} F^{(n)}$ with $F^{(n)} = A^{(n)} \times B^{(n)}$ $\forall n \in \mathbb{N}$

then $\mu(F^{(n)}) < \infty$ $\forall n \in \mathbb{N}$

suppose, $E \in \mathcal{F}_1$, then $E \cap F^{(n)} \uparrow E$, using continuity from below of μ (by continuity)

$$\lim_{n\to\infty} \mu(E \cap F^{(n)}) = \mu(E) \quad \text{---} \quad (3)$$

For each $n \in \mathbb{N}$, apply step (3) for $\mu(E \cap F^{(n)})$

$$\Rightarrow \int_{\Omega_1} \mu_2\left(\left(E \cap F^{(n)}\right)_x\right) d\mu_1(x) = \mu(E \cap F^{(n)}) \quad \text{---} \quad (4)$$

(as $\mu(\Omega \cap F^{(n)}) < \infty$)

$\forall x \in \Omega_1$, $(E \cap F^{(n)})_x \uparrow E_x$

$$\Rightarrow \mu(E \cap F^{(n)})_x \uparrow \mu(E_x)$$

or $x \to \mu(E_x)$ is $\mathcal{F}_1$ measurable

and by monotone conv theorem

$$\lim_{n\to\infty} \int_{\Omega_1} \mu_2(E \cap F^{(n)})_x\, d\mu_1(x) = \int_{\Omega_1} \mu_2(E_x)\, d\mu_1(x)$$

$$= \mu(E \cap F^{(n)}) \ (\because (4))$$

$$\Rightarrow \lim_{n\to\infty} \mu(E \cap F^{(n)}) = \int_{\Omega_1} \mu_2(E_x)\, d\mu_1(x)$$

$$\Rightarrow \mu(E) = \int_{\Omega_1} \mu_2(E_x)\, d\mu_1(x) \quad (\because \textcircled{3})$$

Exe: show that $\mathcal{H}(S) = \mathcal{B}(\mathbb{R}^2)$

Ans: as $\mathcal{B}(\mathbb{R}^2)$ is generated by open sets in $\mathbb{R}^2$

as any open set in $\mathbb{R}^2$ is countable union of rectangles

$$\mathcal{T} = \{u \mid u \text{ is open in } \mathbb{R}^2\} \subseteq \mathcal{H}(S)$$

$$\Rightarrow \mathcal{B}(\mathbb{R}^2) \subseteq \mathcal{H}(S)$$

also, $S \subseteq \mathcal{B}(\mathbb{R}^2)$ is trivial $\Rightarrow \mathcal{H}(S) \subseteq \mathcal{B}(\mathbb{R}^2)$

$$\Rightarrow \mathcal{B}(\mathbb{R}^2) = \mathcal{H}(S)$$

12th March:

Theorem: (Tonelli) Suppose $f: \Omega \to [0,\infty]$ is measurable, then

$$\int_{\Omega_1}\left(\int_{\Omega_2} f_x(y)\,d\mu_2(y)\right)d\mu_1(x) = \int_{\Omega} f\,d\mu$$

and

$$\int_{\Omega_2}\left(\int_{\Omega_1} f^y(x)\,d\mu_1(x)\right)d\mu_2(y) = \int_{\Omega} f\,d\mu$$

$\int_{\Omega} f\,d\mu$ does not have to be $<\infty$ strictly, i.e we don't know if f is integrable

proof:

for $\forall x \in \Omega_1$, we need to show that

$x \to \int f_x(y)\,d\mu_2(y)$ is $\mathcal{F}_1$-measurable and then

prove first identity $\mathcal{F}_2$, now if $f = \chi_E$ for some $E \in \mathcal{F}$, then the statement holds true (∵ prev theorm)

suppose $f = \sum_{i=1}^{n} c_i \chi_{E_i}$ where $c_i > 0$ and $\{E_i\}_{i=1}^{n} \subset \mathcal{F}_1 \otimes \mathcal{F}_2$ i.e it is measurable portion of Ω

then, $\int_{\Omega_2} f_x(y)\,d\mu_2(y) = \sum_{i=1}^{n} c_i \underbrace{\int_{\Omega_2} (\chi_{E_i})_x(y)\,d\mu_2(y)}$

measurable (∵ prev theorm)

$$\Rightarrow x \to \sum_{i=1}^{n} c_i \int_{\Omega_2} (\chi_{E_i})_x(y)\,d\mu_2(y) \text{ is } \mathcal{F}_1\text{-measurable}$$

so, the statement is true for non-negative basic functions

and $\int_{\Omega_2} (\chi_{E_i})_x(y)\,d\mu_2(y) = \mu_2(E_{i_x})$

$\int_{\Omega_1} \mu_2(E_{i_x})\,d\mu_1(x) = \mu(E_i)$ (∵ prev theorm)

as, $\int_{\Omega} f\,d\mu = \sum_{i=1}^{n} c_i\,\mu(E_i)$, theorm statement is true

now, as $f: \Omega \to [0,\infty]$ is a $\mathcal{F}_1$-measurable function, $\exists$ a sequence $\{f_n\}_{n=1}^{\infty}$ of non-negative basic and $\mathcal{F}_1$- measurable functions

s.t $\forall \omega \in \Omega$, $f_n(\omega) \uparrow f(\omega)$ as $n \to \infty$

then, $\forall \omega_2 \in \Omega_2$, $(f_n)_x(\omega_2) \uparrow f_x(\omega_2)$ as $n \to \infty$
(∵ it follows from definition of sections)

by monotone convergence theorm,

$$\lim_{n\to\infty} \int_{\Omega_2} (f_n)_x(y)\,d\mu_2(y) = \int_{\Omega_2} f_x(y)\,d\mu_2(y)$$

also by previous step $x \to \int_{\Omega_2} (f_n)_x(y)\,d\mu_2(y)$ is measurable

as, $\int_{\Omega_2} f_x(y)\,d\mu_2(y)$ is point-wise limit of $\int_{\Omega_2} (f_n)_x(y)\,d\mu_2(y)$

$\Rightarrow x \longrightarrow \int_{\Omega_2} f_x(y)\,d\mu_2(y)$ is measurable

by monotone convergence theorem once again, we get

$$\lim_{n\to\infty} \int_{\Omega_1} \left(\int_{\Omega_2} (f_n)_x(y)\,d\mu_2(y) \right) d\mu_1(x)$$

$$= \int_{\Omega_1} \left(\int_{\Omega_2} f_x(y)\,d\mu_2(y) \right) d\mu_1(x)$$

$$\Rightarrow \lim_{n\to\infty} \int_{\Omega} f_n\,d\mu = \int_{\Omega_1} \left(\int_{\Omega_2} f_x(y)\,d\mu_2(y) \right) d\mu_1(x) \quad (\because f_n \text{ are basic, so from prev step})$$

$$\Rightarrow \int_{\Omega} f\,d\mu = \int_{\Omega_1} \left(\int_{\Omega_2} f_x(y)\,d\mu_2(y) \right) d\mu_1(x) \quad (\because \text{definition of lebesgue integral})$$

Theorem: (Fubini) Suppose $f: \Omega \longrightarrow \overline{\mathbb{R}}$ is integrable, then

$$\int_{\Omega_1} \left(\int_{\Omega_2} f_x(y)\,d\mu_2(y) \right) d\mu_1(x) = \int_{\Omega} f\,d\mu = \int_{\Omega_2} \left(\int_{\Omega_1} f^y(x)\,d\mu_1(x) \right) d\mu_2(y)$$

proof: By the hypothesis,

$$\int_{\Omega} f^+\,d\mu < \infty \text{ and } \int_{\Omega} f^-\,d\mu < \infty \quad (\because \text{definition of integrable functions})$$

by Tonelli's theorem (measurable from prev theorem)

$$x \longrightarrow I^{\pm}(x) = \int_{\Omega_2} (f^{\pm})_x(y)\,d\mu_2(y) < \infty \text{ a.e}$$

let $E^+ = \{x \in \Omega_1 \mid I^+(x) < \infty\} = (I^+)^{-1}([-\infty, \infty)) \in \mathcal{A}_1$ as I^+ is $\mathcal{A}_1$-measurable

$E^- = \{x \in \Omega_1 \mid I^-(x) < \infty\} = (I^-)^{-1}([-\infty, \infty)) \in \mathcal{A}_1$ as I^- is $\mathcal{A}_1$-measurable

let $g^+(x) = \begin{cases} I^+(x) & ; x \in E^+ \\ 0 & ; x \notin E^+ \end{cases}$ g^+ is also measurable

$g^-(x) = \begin{cases} I^-(x) & ; x \in E^- \\ 0 & ; x \notin E^- \end{cases}$ g^- is also measurable

let $g(x) = g^+(x) - g^-(x) \quad \forall x \in \Omega_1$ as $g^+(x) < \infty$, $g^-(x) < \infty \ \forall x \in \Omega_1$

by Tonelli's theorem, $\int_{\Omega_1} g(x)\,d\mu_1(x) = \int_{\Omega_1} g^+(x)\,d\mu_1(x) - \int_{\Omega_1} g^-(x)\,d\mu_1(x)$ ← we don't know if this is well defined or not as $\int = \infty$, $\int = \infty$

$$\int_{\Omega_1} g(x)\,d\mu_1(x) = \int_{\Omega_1} \left(\int_{\Omega_2} (f^+)_x(y)\,d\mu_2(y) \right) d\mu_1(x) - \int_{\Omega_1} \left(\int_{\Omega_2} (f^-)_x(y)\,d\mu_2(y) \right) d\mu_1(x)$$

$$= \int_{\Omega} f^+\,d\mu - \int_{\Omega} f^-\,d\mu = \int_{\Omega} f\,d\mu \quad (\text{we start from } \int_{\Omega} f\,d\mu)$$

12th March :

Theorem: (Fubini) Suppose $f: \Omega \to \overline{\mathbb{R}}$ is integrable, then

$$\int_{\Omega_1}\left(\int_{\Omega_2} f_x(y)\,d\mu_2(y)\right)d\mu_1(x) = \int_{\Omega} f\,d\mu = \int_{\Omega_2}\left(\int_{\Omega_1} f^y(x)\,\mu_1(x)\right)d\mu_2(y)$$

proof: we had seen

$$g^{\pm}(x) = \begin{cases} \int_{\Omega_2} f_x^{\pm}(y)\,d\mu_2(y) & ; x \in E^{\pm} \\ 0 & ; x \notin E^{\pm} \end{cases}$$

then $g = g^+ - g^- : \Omega_1 \to \mathbb{R}$

s.t $\int_{\Omega_1} g(x)\,d\mu_1(x) = \int_{\Omega} f\,d\mu$ ——①

let $E = E^+ \cap E^-$

then, $E^c = (E^+)^c \cup (E^-)^c$

$\Rightarrow \mu_1(E^c) = 0$ as $\mu_1(E^+) = 0$, $\mu_1(E^-) = 0$

then, $\int_{\Omega_1} g(x)\,d\mu_1(x) = \int_{\Omega_1}\left(g^+(x) - g^-(x)\right)d\mu_1(x)$

$$= \int_E (g^+(x) - g^-(x))\,d\mu_1(x) \qquad (\because \mu_1(E^c) = 0)$$

$$= \int_E \left(\int_{\Omega_2} (f_x^+(y) - f_x^-(y))\,d\mu_2(y)\right) d\mu_1(x)$$

$$= \int_{\Omega_1}\left(\int_{\Omega_2} (f_x^+(y) - f_x^-(y))\,d\mu_2(y)\right) d\mu_1(x)$$

$$(\because \mu_1(E^c) = 0)$$

$$= \int_{\Omega_1}\left(\int_{\Omega_2} f_x(y)\,d\mu_2(y)\right) d\mu_1(x) \text{ ——②}$$

So, from ①, ② $\int_{\Omega_1}\left(\int_{\Omega_2} f_x(y)\,d\mu_2(y)\right)d\mu_1(x) = \int_{\Omega} f\,d\mu$

Conv: Suppose $f: \Omega \to \mathbb{R}$ is measurable and

$$\int_{\Omega_1}\left(\int_{\Omega_2} |f_x(y)|\,d\mu_2(y)\right) d\mu_1(x) < \infty$$

then, the conclusion of fubini's theorem hold true

proof:

$0 \le f_x^+ \le |f_x|$

$0 \le f_x^- \le |f_x| \quad \forall\, x \in \Omega_1$

$\Rightarrow \int_{\Omega_1}\left(\int_{\Omega_2} f_x^{\pm}(y)\,d\mu_2(y)\right) d\mu_1(x) < \infty$

then, tonini's theorem,

$$\int_\Omega f^{\pm} d\mu = \int_{\Omega_1}\left(\int_{\Omega_2} f_x^{\pm}(y)\, d\mu_2(y)\right) d\mu_1(x) < \infty$$

$$\Rightarrow \int_\Omega f^{\pm} d\mu < \infty$$

$$\Rightarrow \int_\Omega f\, d\mu = \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu < \infty$$

$$\Rightarrow \int_\Omega f d\mu < \infty \quad \text{or } f \text{ is integrable}$$

then, we can apply fubini theorem

Theorm: let $(X, \mathcal{H}, \nu)$ be any σ-finite measure space and let $f: X \longrightarrow [0, \infty)$ consider the product space, s.t f is measurable

$$(X \times \mathbb{R}, \mathcal{H} \otimes \mathcal{B}(\mathbb{R}), \nu \otimes \lambda)$$

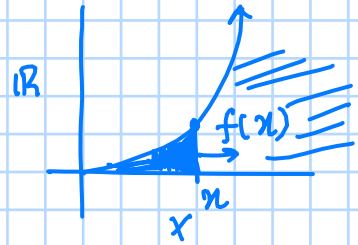
suppose $S = \{(x, y) \in X \times \mathbb{R} \mid 0 \le y \le f(x)\}$

then $S \in \mathcal{H} \otimes \mathcal{B}(\mathbb{R})$ and $\int_X f(x)\, d\nu(x) = (\nu \otimes \lambda)(S)$

proof: $\phi_1, \phi_2, \psi : X \times \mathbb{R} \longrightarrow \mathbb{R}$ by $\phi_1(x, y) = f(x)$

$$\phi_2(x, y) = -y$$

$$\psi(x, y) = \phi_1(x, y) + \phi_2(x, y)$$



let $I \subseteq \mathbb{R}$ be an interval ($\because I \in \mathcal{B}(\mathbb{R})$)

then, $\phi_1^{-1}(I) = f^{-1}(I) \times \mathbb{R} \in \mathcal{H}_1 \otimes \mathcal{B}(\mathbb{R})$

($\because f$ is measurable)

$$\phi_2^{-1}(I) = X \times (-I) \in \mathcal{H} \otimes \mathcal{B}(\mathbb{R})$$

$\Rightarrow \phi_1, \phi_2$ is measurable

$\Rightarrow \psi$ is measurable

then $\psi^{-1}([0, \infty)) \cap (X \times [0, \infty))$ (both measurable)

$$= \{(x, y) \in X \times \mathbb{R} \mid f(x) - y \ge 0\} \cap (X \times [0, \infty))$$

$$\in \mathcal{H} \otimes \mathcal{B}(\mathbb{R}) \qquad \in \mathcal{H} \otimes \mathcal{B}(\mathbb{R})$$

$\Rightarrow S = \psi^{-1}([0, \infty)) \cap (X \times [0, \infty))$ is measurable

then, $(\nu \otimes \lambda)(S) = \int_\Omega \chi_S\, d(\nu \otimes \lambda)$; $\Omega = X \times \mathbb{R}$

$$= \int_X \left(\int_{\mathbb{R}} (\chi_S)_x(y)\, d\lambda(y)\right) d\nu(x) \quad (\because \text{fubini's theorm})$$

$$= \int_X \left(\int_{\mathbb{R}} \chi_{S_x}(y)\, d\lambda(y) \right) d\nu(x)$$

$$= \int_X \left(\int_{\mathbb{R}} \chi_{\{\tilde{y}\in\mathbb{R} \mid (x,\tilde{y})\in S\}}(y)\, d\lambda(y) \right) d\nu(x)$$

$$= \int_X \left(\int_{\mathbb{R}} \chi_{[0,f(x)]}(y)\, d\lambda(y) \right) d\nu(x)$$

$$= \int_X f(x)\, d\nu(x)$$

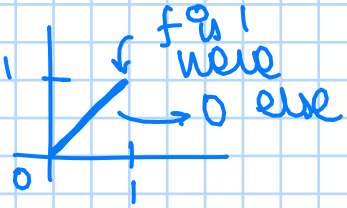
Eg: Let $\Omega_i = [0,1]$ $\mathcal{F}_i = \mathcal{B}([0,1])$ $\forall i = 1,2$

and then take $\mu_1 = \lambda$ $\mu_2 = C$ (counting measure)

$([0,1], \mathcal{B}([0,1]), C)$ is not σ-finite

Let $f : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R}$ by

$$f = \chi_{\{(x,y)\in \Omega_1\times\Omega_2 \mid x=y\}}$$



f is measurable if diagonal is measurable

as, $n\in\mathbb{N}$, $I_j^n = \left[\frac{j-1}{n}, \frac{j}{n}\right]$, $1\le j\le n$

$$D_n = \bigcup_{j=1}^{n} (I_j^n \times I_j^n) \in \mathcal{B}([0,1]) \otimes \mathcal{B}([0,1])$$

$$\Rightarrow \bigcap_{n=1}^{\infty} D_n \in \mathcal{B}([0,1]) \otimes \mathcal{B}([0,1])$$

$$\Rightarrow \{(x,y)\in\Omega_1\times\Omega_2 \mid x=y\} \in \mathcal{B}([0,1]) \otimes \mathcal{B}([0,1])$$

So, f is measurable

consider, $\int_{\Omega_2} f(x,y)\, d\,C(y) = 1$ ($\because$ x is fixed)

$C(\{x\}) = 1$

$$\Rightarrow \int_{\Omega_1} \left(\int_{\Omega_2} f(x,y)\, dC(y) \right) d\lambda(x) = \int_{\Omega_1} 1\, d\lambda(x) = 1$$

$$\int_{\Omega_1} f(x,y)\, d\lambda(x) = 0 \Rightarrow \int_{\Omega_2} \left(\int_{\Omega_1} f(x,y)\, d\lambda(x) \right) d\,C(y) = 0 \neq 1$$

as measure of set with $x=y$ is $\{x\}$, $\lambda(\{y\}) = 0$ (fixed)

Note: Above is a counterexample, which shows that Fubini's theorem fails when we drop σ-additivity

Eg: Let's say we don't have integrability

$\Omega_1 = [0,1]$ $\mathcal{F}_i = \mathcal{B}([0,1])$ $\mu_i = \lambda$ $\forall i = 1,2$

let $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2} & ; \text{if } (x,y) \neq 0 \\ 0 & ; \text{if } (x,y) = 0 \end{cases}$$

as $\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) = f(x,y) = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$

$$\int_0^1 f(x,y)\, d\lambda(y) = \frac{1}{1+x^2}$$

$$\int_0^1 f(x,y)\, d\lambda(x) = \frac{-1}{1+y^2}$$

$$\Rightarrow \int_0^1 \int_0^1 f(x,y)\, d\lambda(y)\, d\lambda(x) = \pi/4 \qquad (\because \text{Riemann is Lebesgue for bounded})$$

but $$\int_0^1 \int_0^1 f(x,y)\, d\lambda(x)\, d\lambda(y) = -\pi/4$$

$\therefore$ Fubini fails

as if f is integrable
then f^+, f^- integrable $\Rightarrow |f|$ integrable

but $$\int_{[0,1]\times[0,1]} |f(x,y)|\, d(\lambda \times \lambda) = \int_0^1 \left(\int_0^1 |f(x,y)|\, d\lambda(y) \right) d\lambda(x)$$

($\because$ Tolivii theorem)

$$\geq \int_0^1 \int_0^x |f(x,y)|\, d\lambda(y)\, d\lambda(x)$$

(always ≥ 0)

$$= \int_0^1 \int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\, d\lambda(y)\, d\lambda(x)$$

$y = tx$ then

$$= \int_0^1 \frac{1}{x} \left(\int_0^1 \frac{1-t^2}{(1+t^2)^2}\, d\lambda(t) \right) d\lambda(x)$$

($\because$ everything is in Riemann so we can do change of variable)

$$= c \int_0^1 \frac{1}{x}\, d\lambda(x) = \infty$$

$c > 0$

$\therefore |f|$ not integrable $\Rightarrow f$ is not integrable

16th March:

Now, we will be seeing application part of measure theory, it will be useful in functional analysis

L^p-spaces:

Throughout we will fix $p \in [1,\infty]$ and $p' \in [1,\infty]$ s.t

$$\frac{1}{p} + \frac{1}{p'} = 1$$

p' is the dual of p

Eg: $p=1 \Rightarrow p'=\infty$
$p'=1 \Rightarrow p=\infty$
$p=2,\ p'=2$

Lemma: Suppose $a,b \in \mathbb{C}$ and $p \geq 1$, then

(i) $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$

(ii) $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$ $\forall\ a,b \geq 0$ and $p \geq 1$, is young's inequality

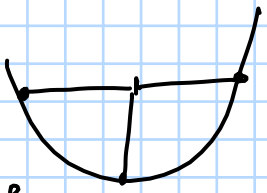
proof: (i) Consider $f(x) = x^p$ for $x > 0$, then

f is non-negative and $f'(x) = px^{p-1} > 0$

$f''(x) = \underbrace{p(p-1)}_{\geq 0} \underbrace{x^{p-2}}_{\geq 0}$

$\Rightarrow f''(x) \geq 0$

$f(x)$ is a convex function



$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \quad \forall \lambda \in [0,1] \quad \forall x, y > 0$$

now, $\left|\frac{a+b}{2}\right|^p = f\left(\left|\frac{a}{2} + \frac{b}{2}\right|\right)$

$\leq f\left(\frac{|a|}{2} + \frac{|b|}{2}\right)$ ($\because f$ is increasing)

$\leq \frac{1}{2} f(|a|) + \frac{1}{2} f(|b|)$

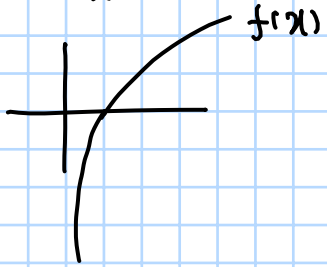
$\Rightarrow |a+b|^p \leq 2^p \left[\frac{1}{2}|a|^p + \frac{1}{2}|b|^p\right]$

$\Rightarrow |a+b|^p \leq 2^{p-1}|a|^p + 2^{p-1}|b|^p$

(ii) consider $f(x) = \log x$, $x > 0$ then

$f'(x) = \frac{1}{x} > 0$

$f''(x) = -\frac{1}{x^2} < 0$



f is concave function

for $p \neq \infty$, $p' \neq \infty$:

So, $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y)$ $\forall$ $\lambda \in [0,1]$, $x, y > 0$

then $\log\left(\frac{a^p}{p} + \frac{b^{p'}}{p'}\right) \geq \frac{1}{p}\log(a^p) + \frac{1}{p'}\log(b^{p'})$

$= \log(a) + \log(b)$

$= \log(ab)$

$\Rightarrow \log\left(\frac{a^p}{p} + \frac{b^{p'}}{p'}\right) \geq \log(ab)$

$\Rightarrow \frac{a^p}{p} + \frac{b^{p'}}{p'} \geq ab$ ($\because f$ is increasing)

Exe: show Young's inequality for $p = \infty$ or $p' = \infty$ → done down

Def: suppose $(\Omega, \mathcal{H}, \mu)$ is a measure space, $f: \Omega \to \mathbb{C}$ is a measurable function and $p \in (0, \infty)$, then

$$\|f\|_p = \left(\int_\Omega |f|^p \, d\mu\right)^{\frac{1}{p}}$$

Note: $\|f\|_p = 0 \Rightarrow f = 0$ a.e

for $\Omega = \{1, 2 \dots n\}$
$\mathcal{H} = \mathcal{P}(\Omega)$
$\mu = c$

then $\|f\|_p = \left(\sum_{i=1}^{n} |f(i)|^p\right)^{\frac{1}{p}}$, then $\{f: \Omega \to \mathbb{C}\} = \{(x_k)_{k=1}^{n} \mid x_k \in \mathbb{C}\}$

$\lim_{p \to \infty} \|f\|_p = \max_{1 \leq i \leq n} \{|f(i)|\}$, this is true case as $\mu = c$ and it is finite

Theorem: (Holder's inequality) suppose $p \in (1, \infty)$ and $f, g: \Omega \to \mathbb{C}$ are measurable functions, then

$$\int_\Omega |f(\omega)\, g(\omega)| \, d\mu(\omega) \leq \|f\|_p \, \|g\|_{p'}$$

i.e $\|fg\|_1 \leq \|f\|_p \|g\|_{p'}$

proof:

the inequality is trivial if $\|f\|_p = 0$ or $\|g\|_{p'} = 0$

or $\|f\|_p = \infty$ or $\|g\|_{p'} = \infty$

So, we have to show this statement for $\|f\|_p, \|g\|_{p'} \in (0, \infty)$

suppose, $\|f\|_p = 1 = \|g\|_{p'}$, ($\because$ we can normalise any $\|f_p\| < \infty$)

for $\omega \in \Omega$, $|f(\omega) g(\omega)| \leq \frac{|f(\omega)|^p}{p} + \frac{|g(\omega)|^{p'}}{p'}$ ($\because$ Young's inequality)

$$\Rightarrow \int_\Omega |f(\omega) g(\omega)| \, d\mu(\omega) \leq \int_\Omega \left(\frac{|f(\omega)|^p}{p}\right) d\mu(\omega) + \int_\Omega \left(\frac{|g(\omega)|^{p'}}{p}\right) d\mu(\omega)$$

$$\Rightarrow \int_\Omega |f(\omega)g(\omega)|\,d\mu(\omega) \le \frac{\|f\|_p^p}{p} + \frac{\|g\|_{p'}^{p'}}{p'}$$

$$\Rightarrow \int_\Omega |f(\omega)g(\omega)|\,d\mu(\omega) \le 1 = \|f\|_p \|g\|_p \quad \text{—— ①}$$

for $f, g : \Omega \to \mathbb{C}$ s.t $\|f\|_p, \|g\|_{p'} \in (0,\infty)$

let $F(\omega) = \frac{f(\omega)}{\|f\|_p}$ $\quad \|F\|_p = 1$

$G(\omega) = \frac{g(\omega)}{\|g\|_{p'}}$ $\quad \|G\|_p = 1$

then, $\int_\Omega |F(\omega)G(\omega)|\,d\mu(\omega) \le \|F\|_p \|G\|_p = 1$ $\quad (\because ①)$

$$\Rightarrow \int_\Omega \left| \frac{f(\omega)g(\omega)}{\|f\|_p \|g\|_{p'}} \right| d\mu(\omega) \le 1$$

$$\Rightarrow \int_\Omega |f(\omega)g(\omega)|\,d\mu(\omega) \le \|f\|_p \|g\|_{p'}$$

∴, $\|fg\|_1 \le \|f\|_p \|g\|_{p'}$ for $p \in (1,\infty)$

Defn: for $1 \le p < \infty$, let

$$\mathcal{L}^p(\Omega, \mathcal{H}, \mu) = \{ f : \Omega \to \mathbb{C} \mid f \text{ is measurable}, \|f\|_p < \infty \}$$

observe that for $f, g \in \mathcal{L}^p(\Omega, \mathcal{H}, \mu)$ $\|f\|_p = 0 = \|g\|_p \not\Rightarrow f = g$

∴, let $L^p(\Omega, \mathcal{H}, \mu) = \mathcal{L}^p(\Omega, \mathcal{H}, \mu)/\sim$ where $f \sim g$ if $f = g$ a.e
then, this makes $\|\cdot\|_p$ a valid norm on $L^p(\Omega, \mathcal{H}, \mu)$

we will use $L^p(\Omega, \mathcal{H}, \mu)$ by $L^p(\Omega)$ when $\mathcal{H}, \mu$ are clearly defined above

Note: $L^p(\Omega)$ is a Banach vector space as $f, g \in L^p(\Omega)$, $f + g \in L^p(\Omega)$ $\quad (\because$ Young's inequality)

Exe: under what condition $\dim(L^p(\Omega)) < \infty$, $|\Omega| = \inf$, $|\Omega| =$ finite → leave down

Theorem: (Minkowski's inequality) If $1 \le p < \infty$, $f, g \in L^p(\Omega)$, then

$\|f + g\|_p \le \|f\|_p + \|g\|_p$, triangle inequality for p-norm

proof:

If $f + g = 0$ a.e then $\|f + g\|_p = 0$

also if $p = 1$ then $|f(\omega) + g(\omega)| \le |f(\omega)| + |g(\omega)|$

$\Rightarrow \|f + g\|_1 \le \|f\|_1 + \|g\|_1$

∴, let $p > 1$ and $f + g \neq 0$ a.e

then $\frac{1}{p} + \frac{1}{p'} = 1 \Rightarrow (p-1)p' = p$

using triangle inequality

$$|f(w)+g(w)|^p \le |f(w)+g(w)|^p \left(|f(w)|+|g(w)|\right) \quad \forall w \in \Omega$$

then, $\int_\Omega |f(w)+g(w)|^p d\mu(w)$

$$\le \int_\Omega |f(w)|\,|f(w)+g(w)|^{p-1} d\mu(w) + \int_\Omega |g(w)|\,|f(w)+g(w)|^{p-1} d\mu(w)$$

by Holder's inequality

$$\int_\Omega |f(w)+g(w)|^p d\mu(w) \le \|f\|_p \left[\left(\int_\Omega |f(w)+g(w)|^{(p-1)(p')} d\mu(w)\right)^{\frac{1}{p'}}\right] + \|g\|_p \left[\left(\int_\Omega |f(w)+g(w)|^{(p-1)p'} d\mu(w)\right)^{\frac{1}{p'}}\right]$$

$$\le \left(\|f\|_p + \|g\|_p\right)\left(\left(\int_\Omega |f(w)+g(w)|^p d\mu(w)\right)^{\frac{1}{p'}}\right)$$

$$\Rightarrow \left(\int_\Omega |f(w)+g(w)|^p d\mu(w)\right)^{1-\frac{1}{p'}} \le \|f\|_p + \|g\|_p$$

$$\Rightarrow \|f+g\|_p \le \|f\|_p + \|g\|_p$$

Note: from above inequality we get $\|\cdot\|_p : L^p \to [0,\infty)$ is a norm

Exe: show Young's inequality for $p=\infty$ or $p'=\infty$

Ans:

wog as $\frac{1}{p}+\frac{1}{p'}=1$ let $p=1$ and $p'=\infty$

then $\forall a,b>0$ in $\mathbb{R}$

if $b>1$, then $\frac{b^{p'}}{p'} \to \infty$ $(\because p' \to \infty)$

$\Rightarrow ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}$ is trivial

then, if $b \in [0,1]$ then

$\frac{b^{p'}}{p'} \to 0$ $(\because p' \to \infty)$

now as $b \in [0,1]$, $ab \le a = \frac{a^p}{p} \le \frac{a^p}{p} + 0 = \frac{a^p}{p} + \frac{b^{p'}}{p'}$

so, we get the inequality for p or $p'=\infty$

Exe: under what condition $\dim(L^p(\Omega)) < \infty$, $|\Omega| = \inf$, $|\Omega| =$ finite

Ans:

$|\Omega| =$ finite then $\mathcal{L}^p(\Omega) = \{f : \Omega \to \mathbb{C} \mid f$ is measurable, $\|f\|_p < \infty\}$

Ω finite $\Rightarrow f$ can be written as

$(f(x_1), f(x_2) \ldots f(x_N))$ (finite)

finite $\{x_1 \ldots x_N\} = \Omega$

dimension

so, $L^p(\Omega)$ dim is finite

also $|\Omega| = \inf$ if $\mu \equiv 0$ then, $\|f\|_p = 0$, f is measurable if $f^{-1}(F) \in \{\emptyset, \Omega\}$

$\mathcal{F} = \{\emptyset, \Omega\}$

then $\mathcal{L}^p(\Omega) = \{f : \Omega \to \mathbb{C}\}$

but $f : \Omega \to \mathbb{C} \Rightarrow f \equiv C$ some constant $C \in \mathbb{C}$

so dim = 1

23rd March:

$(\Omega, \mathcal{A}, \mu)$-measure space, $1 \le p < \infty$ $L^p(X)$ is a vector space over $\mathbb{C}$

$\|\cdot\|_p : L^p(X) \to \mathbb{R}$ defines a norm on $L^p(X)$

Equivalently, $\|f\|_p = 0 \Leftrightarrow f = 0$

$\|cf\|_p = |c| \|f\|_p \quad \forall c \in \mathbb{C}, f \in L^p(X)$

$\|f+g\|_p \le \|f\|_p + \|g\|_p \quad \forall f, g \in L^p(X)$ ($\because$ Minkowski inequality)

$\|\cdot\|_p$ induces a metric on $L^p(X)$

$d_p(f,g) = \|f-g\|_p \quad \forall f, g \in L^p(X)$

So, $(L^p(X), d_p)$ is a metric space

Note: $(L^p(X), \|\cdot\|_p)$ is a normed linear space

Defn: A metric space (X,d) is said to be complete if for every Cauchy sequence in (X,d) is convergent in (X,d)

to show $\mathbb{C}^n$ is complete, we show component wise Cauchy $\Rightarrow$ Conv component wise $\Rightarrow$ so we do it for $\mathbb{C}^n$

Eg: $X = \mathbb{N}$, $\mathcal{A} = \mathcal{P}(\mathbb{N})$ $\mu = c$, then

$$\ell^p(\mathbb{N}) = \left\{ (x_n)_{n=1}^{\infty} \;\middle|\; \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

suppose, $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\ell^p(\mathbb{N})$

where $f_n : \mathbb{N} \to \mathbb{R}$

$\{f_n(k)\}_{n=1}^{\infty}$ is also Cauchy $\forall k \in \mathbb{N}$

As $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall m, n \ge N$

$$\|f_n - f_m\|_p < \varepsilon$$

$$\Rightarrow \left(\sum_{k=1}^{\infty} |f_m(k) - f_n(k)|^p \right)^{\frac{1}{p}} < \varepsilon$$

$$\Rightarrow |f_m(k) - f_n(k)| < \varepsilon \quad \forall k \in \mathbb{N}$$

So, $\{f_n(k)\}_{k=1}^{\infty}$ is a Cauchy seq in $\mathbb{C}$ $\forall k \in \mathbb{N}$

Let $f : \mathbb{N} \to \mathbb{C}$ by $f(k) = \lim_{n \to \infty} f_n(k)$ $\forall k \in \mathbb{N}$

(pointwise limit)

Note: $f \in \ell^p(\mathbb{N})$ and $\|f_n - f\|_p \to 0$ as $n \to \infty$ is from tutorial 9

Note: Above, the pointwise limit converge we got above, but there can be cases when we cannot do this

Eg: consider the following situation:

$\Omega = [0,1]$

$\mathcal{F} = \mathcal{B}([0,1])$ $\mu = \lambda$

fixed $p \in [1,\infty)$ let $\{f_n\}_{n=1}^{\infty} \subseteq L^p(\Omega)$ by

$f_1 = \chi_{[0,1]}$

$f_2 = \chi_{[0,\frac{1}{2})}$

$f_3 = \chi_{[\frac{1}{3},1]}$

$f_4 = \chi_{[0,\frac{1}{3})}$

$f_5 = \chi_{[\frac{1}{3},\frac{2}{3})}$

$f_6 = \chi_{[\frac{2}{3},1]}$

$\vdots$

now, $\|f_n\|_p \longrightarrow 0$ as $n \longrightarrow \infty \Rightarrow \{f_n\}_{n=1}^{\infty}$ is convergent

$\therefore$, $\{f_n\}_{n=1}^{\infty}$ is cauchy

but observe that for a fixed $x \in [0,1]$, there are infinitely many seqes s.t $f_n(x) = 1$ and infinitely many seq s.t $f_n(x) = 0$

$\therefore$, $\{f_n(x)\}_{n=1}^{\infty}$ does not converge $\forall x \in [0,1]$

1) If a subsequence $\{x_{n_k}\}$ of a cauchy sequence $\{x_n\}$ in a metric space (X,d) converge to $x_0 \in X$, then

$$x_n \to x_0 \text{ as } n \to \infty \text{ in } (X,d)$$

2) to show the completeness of (X,d) it is sufficient to check that a cauchy seq $\{x_n\}_{n=1}^{\infty}$ satisfying $d(x_{n+1}, x_n) < \frac{1}{2^n}$ $\forall n \in \mathbb{N}$ is convergent

Note: every cauchy sequence has a subsequence satisfying (2)

Exe: show the two statements above → done down

Theorm: $\forall p \in [1,\infty)$, $L^p(\Omega)$ is a complete metric space

proof: WLOG, let $\{f_n\}_{n=1}^{\infty} \subseteq L^p(\Omega)$ be a cauchy sequence s.t

$$\|f_{n+1} - f_n\|_p < \frac{1}{2^n} \quad \forall n \in \mathbb{N}$$

let $g(w) = \sum_{n=1}^{\infty} |f_{n+1}(w) - f_n(w)|$ and $g_k(w) = \sum_{n=1}^{k} |f_{n+1}(w) - f_n(w)|$

$\forall w \in \Omega$, $k \in \mathbb{N}$

if we show $g \in L^p(\Omega)$, we are done

$$\|g_k\|_p \le \sum_{n=1}^{k} \|f_{n+1} - f_n\|_p < \sum_{n=1}^{k} \frac{1}{2^n} \le 1 \quad \forall k \in \mathbb{N}$$

so, $g_k \in L^p(\Omega)$

$g_k(\omega) \uparrow g(\omega)$ we apply monotone conv. theorem

$$\Rightarrow \int_\Omega |g(\omega)|^p d\mu(\omega) = \lim_{k\to\infty} \int_\Omega |g_k(\omega)|^p d\mu(\omega)$$

$$= \lim_{k\to\infty} \|g_k\|_p^p \le 1$$

$$\Rightarrow \|g\|_p^p \le 1 \Rightarrow \|g\|_p \le 1$$

$$\Rightarrow g \in L^p(\Omega)$$

so, $\exists E \in \mathcal{F}$ s.t. $\mu(E^c) = 0$ and

$$g(\omega) < \infty \quad \forall \omega \in E$$

now, let $h: \Omega \to \mathbb{C}$ by $h(\omega) = \begin{cases} \sum_{n=1}^{\infty} [f_{n+1}(\omega) - f_n(\omega)] & ; \omega \in E \\ 0 & ; \text{otherwise} \end{cases}$

as $g \in L^p(\Omega) \Rightarrow h \in L^p(\Omega)$

$\forall \omega \in \Omega$, $h(\omega) = \lim_{k\to\infty} \sum_{n=1}^{k} (f_{n+1}(\omega) - f_n(\omega))$

$$= \lim_{k\to\infty} f_{k+1}(\omega) - f_1(\omega)$$

$$\Rightarrow f_{k+1}(\omega) \xrightarrow{\text{a.e}} h(\omega) + f_1(\omega) \quad \forall \omega \in \Omega$$

now, let $f = h + f_1$, we have to show 2 things

1) $f \in L^p(\Omega)$
2) $\|f_n - f\|_p \to 0$ as $n \to \infty$

now, as $|h(\omega)| \le g(\omega)$

fixing $\omega \in \Omega$ a.e

and $g \in L^p(\Omega)$, so, $h \in L^p(\Omega)$

$$\Rightarrow f \in L^p(\Omega) \quad \text{——— ①}$$

now, for $m > n$

$$f_m(\omega) - f_n(\omega) = \sum_{k=n}^{m-1} f_{k+1}(\omega) - f_k(\omega)$$

$$\Rightarrow \|f_m(\omega) - f_n(\omega)\|_p \le \sum_{k=n}^{m-1} \|f_{k+1} - f_k\|_p < \sum_{k=n}^{m-1} \frac{1}{2^k} \le \frac{1}{2^{n-1}}$$

$\leftarrow$ taking p

$$\Rightarrow \int_\Omega |f_m(\omega) - f_n(\omega)|^p d\mu(\omega) \le \frac{1}{2^{(n-1)p}}$$

for fixed $n \in \mathbb{N}$, we apply fatou's lemma to get

$$\int_\Omega \liminf_{m\to\infty} |f_m(\omega)-f_n(\omega)|^p d\mu(\omega) \le \liminf_{m\to\infty} \int_\Omega |f_m(\omega)-f_n(\omega)|^p d\mu(\omega)$$

$$\le \frac{1}{2^{(n-1)p}}$$

$$\Rightarrow \int_\Omega |f(\omega)-f_n(\omega)|^p d\mu(\omega) \le \frac{1}{2^{(n-1)p}}$$

$$\Rightarrow \|f-f_n\|_p \le \frac{1}{2^{n-1}}$$

$$\Rightarrow f_n \to f \text{ as } n\to\infty$$

Exe: show the two statements above

Ans: $\{x_{n_k}\}$ subseq conv to x_0

then $|x_{n_k}-x_0|<\varepsilon \ \forall \varepsilon>0, \exists N$ s.t $\forall n_k \ge N$

$\Rightarrow |x_n - x_{n_k}| + |x_{n_k}-x_0| < 2\varepsilon \Rightarrow |x_n - x_0| < 2\varepsilon$

now if cauchy seq s.t $d(x_n, x_{n+1}) \le \frac{1}{2^n} \Rightarrow$ convergent is sufficient as every cauchy seq will have such a subsequence from definition then from first part we get this

26th March:

$L^\infty(\Omega)$:

Defn: Suppose $f:\Omega \to \mathbb{C}$ is a measurable function, then f is said to be essentially bounded if $\exists M>0$ s.t

$$\mu\left(\{\omega\in\Omega \mid |f(\omega)|>M\}\right)=0$$

Defn: $L^\infty((\Omega,\mathcal{F},\mu)) = \{f:\Omega\to\mathbb{C} \mid f \text{ is essentially bounded}\}/\sim$

$f\sim g$ iff $f-g=0$ a.e

Defn: $\|\cdot\|_\infty : L^\infty(\Omega)\to[0,\infty)$ by $\|f\|_\infty = \inf\{M \mid \mu(\{\omega\in\Omega \mid |f(\omega)|>M\})=0\}$

Note: $L^\infty(\Omega) = L^\infty((\Omega,\mathcal{F},\mu))$

From definition, we see that

(1) the inf in definition of $\|\cdot\|_\infty$ is always attained as it sufficient to check that

$$\mu\left(\{\omega\in\Omega \mid |f(\omega)|>\|f\|_\infty\}\right)=0 \quad \text{---} (1)$$

now, $\{\omega\in\Omega \mid |f(\omega)|>\|f\|_\infty\} = \bigcup_{n=1}^{\infty}\{\omega\in\Omega \mid |f(\omega)|>\|f\|_\infty+\frac{1}{n}\}$

fix $n_0\in\mathbb{N}$, then $\exists M>0$ s.t

$0\le M<\|f\|_\infty+\frac{1}{n_0}$ satisfying ($\because$ definition of inf)

$\Rightarrow \mu(\{\omega\in\Omega \mid |f(\omega)|>M\})=0$

and $\{\omega\in\Omega \mid |f(\omega)|>\|f\|_\infty+\frac{1}{n_0}\}\subseteq\{\omega\in\Omega \mid |f(\omega)|>M\}$

$\Rightarrow \mu\left(\{\omega\in\Omega \mid |f(\omega)|>\|f\|_\infty+\frac{1}{n_0}\}\right)=0$

$\Rightarrow \mu(\{\omega\in\Omega \mid |f(\omega)|>\|f\|_\infty\}) \le 0$

So, (1) is satisfied and inf is attained

(2) Suppose $f\in L^\infty(\Omega)$ then, $|f(\omega)|\le\|f\|_\infty$ a.e

So, $\|f\|_\infty=0 \Rightarrow f=0$ a.e

(3) $\|\cdot\|_\infty$ satisfies the triangle inequality,

for $f,g\in L^\infty(\Omega)$

$$|f(\omega)+g(\omega)| \le |f(\omega)|+|g(\omega)|$$
$$\le \|f\|_\infty+\|g\|_\infty \text{ a.e}$$

$\Rightarrow f+g\in L^\infty(\Omega)$

$\Rightarrow \|f+g\|_\infty \le \|f\|_\infty+\|g\|_\infty$

($\because$ inf definition, $M=\|f\|_\infty+\|g\|_\infty$

as $\mu(\{\omega\in\Omega \mid |f(\omega)+g(\omega)|>\|f\|_\infty+\|g\|_\infty\})=0$)

so, from (1), (2), (3) and also $\|c f\|_\infty = |c| \|f\|_\infty$ ($\because$ int also scales)

$\Rightarrow \|\cdot\|_\infty$ is norm for $L^\infty(\Omega)$

(4) it is possible to extend holder's inequality for $p=\infty, q=1$

suppose, $f \in L^\infty(\Omega)$ and $g \in L^1(\Omega)$ then

$$\int_\Omega |fg(w)| d\mu(w) \leq \|f\|_\infty \int_\Omega |g(w)| d\mu(w) = \|f\|_\infty \|g\|_1$$

(5) $d_\infty(f,g) = \|f-g\|_\infty \; \forall f,g \in L^\infty(\Omega)$ is metric on $L^\infty(\Omega)$

THEOREM: $(L^\infty(\Omega), d_\infty)$ is a complete metric

proof: suppose $\{f_n\}_{n=1}^\infty \subseteq L^\infty(\Omega)$ is a cauchy sequence

fix $k \in \mathbb{N}$, then $\exists M_k \in \mathbb{N}$ s.t $\forall m,n \geq N_k$

$$d_\infty(f_n, f_m) = \|f_m - f_n\|_\infty < \frac{1}{k}$$

$$\Rightarrow |f_m(w) - f_n(w)| < \frac{1}{k} \quad a.e$$

so, $\exists E_k \in \mathcal{F}$ s.t $\mu(E_k) = 0$

$$\forall w \in E_k^c \quad |f_m(w) - f_n(w)| < \frac{1}{k}$$

now, let $E = \bigcup_{k=1}^\infty E_k$, then

$$\mu(E) = 0 \quad (\because \mu(E_k) = 0 \; \forall \; k \in \mathbb{N})$$

and $\forall w \in E^c = \bigcap_{k=1}^\infty E_k^c$

$\{f_n(w)\}_{n=1}^\infty$ is a cauchy sequence

as $\forall \varepsilon > 0, \exists k \in \mathbb{N}$ s.t $\frac{1}{k} < \varepsilon \Rightarrow |f_n(w) - f_m(w)| < \frac{1}{k} < \varepsilon$

$\Rightarrow \{f_n(w)\}$ is convergent $\forall w \in E^c$

now, let $f(w) = \begin{cases} \lim_{n\to\infty} f_n(w) & ; w \in E^c \\ 0 & ; w \in E \end{cases}$

f is measurable, as it's pointwise limit of measurable functions

$\forall w \in E^c, \forall n \geq N_k \quad |f(w) - f_n(w)| < \frac{1}{k} \quad a.e$

$$\Rightarrow \|f - f_n\|_\infty < \frac{1}{k} \; \forall n \geq N_k \; \forall k \in \mathbb{N}$$

$$\Rightarrow f_n \to f \text{ in } \|\cdot\|_\infty \text{ n}$$

and $\|f\|_\infty - \|f_n\|_\infty < \frac{1}{k} \Rightarrow \|f\|_\infty < \frac{1}{k} + \|f_n\|_\infty$

$$\Rightarrow |f(w)| < \frac{1}{k} + \|f_n\|_\infty \; a.e$$

so, $f \in L^\infty(\Omega)$ ($\because$ previous part

Dense subsets of L^p:

Consider $f: \Omega \to \mathbb{C}$, then $\exists$ sequences $\{\psi_{i,n}\}_{n=1}^{\infty}$ $\forall i=1,2,3,4$ of non-negative basic functions s.t

$\psi_{1,n} \uparrow Re(f)^+$

$\psi_{2,n} \uparrow Re(f)^-$

$\psi_{3,n} \uparrow Im(f)^+$

$\psi_{4,n} \uparrow Im(f)^-$

then, $\psi_n = (\psi_{1,n} - \psi_{2,n}) + i(\psi_{3,n} - \psi_{4,n})$ is a complex valued basic function

moreover, $0 \leq |\psi_1| \leq |\psi_2| \leq |\psi_3| \leq \cdots \leq |f|$ pointwise

Note: Above $0 \leq |\psi_1| \leq \cdots \leq |f|$ is from tutorial 8

Theorem: If $p \in [1,\infty]$ then $S = \left\{ \sum_{j=1}^{\infty} c_j \chi_{E_j} \;\middle|\; \mu(E_j) < \infty,\ c_j \in \mathbb{C},\ \forall j=1,2\ldots \right\}$ is a dense subspace of $L^p(\Omega)$

proof: Observe that

$S \subseteq L^p(\Omega)$ is a subspace

now, $f \in L^p(\Omega)$, $\exists$ a sequence $\{\psi_n\}_{n=1}^{\infty} \subset S$ s.t $\psi_n(\omega) \to f(\omega)$ as $n \to \infty$ a.e

and then, also $|\psi_n - f|^p \leq (|\psi_n| + |f|)^p \leq 2^p |f|^p$

$\Rightarrow |\psi_n - f|^p \leq 2^p |f|^p$

by DCT: $\psi_n \to f$ in p-norm $p \in [1,\infty)$

Ex: Complete above theorem for $p=\infty$ → done done

now, let $\Omega = \mathbb{R}^n$, $\mathcal{A} = \mathcal{B}(\mathbb{R}^n)$ $\mu = \lambda_n$, and let

$f: \mathbb{R}^n \to \mathbb{C}$ is a continious function and

$\text{supp}(f) = \overline{\{x \in \mathbb{R}^n \mid f(x) \neq 0\}}$

Note: By def support is always a closed set

$C_c(\mathbb{R}^n) = \{ f: \mathbb{R}^n \to \mathbb{C} \mid f \text{ is cont, supp}(f) \text{ is compact} \}$

then $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $p \in [1,\infty)$

Step-1: Suppose $K \subseteq \mathbb{R}^n$ is compact, and $U \subseteq \mathbb{R}^n$ with $K \subseteq U$, then $\exists$ $f \in C_c(\mathbb{R}^n)$ s.t $\text{supp}(f) \subset U$ and $f(x) = 1$ $\forall x \in K$

Step-2: Regularity of λ_n: suppose $E \in \mathcal{B}(\mathbb{R}^n)$, $\forall \varepsilon > 0$, then $\exists$ $K \subseteq \mathbb{R}^n$ compact and $U \subseteq \mathbb{R}^n$ open s.t $K \subset E \subset U$ and

$$\lambda_n(E \setminus K) < \varepsilon \text{ and } \lambda_n(U \setminus E) < \varepsilon$$

then to show this is dense

Step 1: it is sufficient to show that given any $E \in \mathcal{B}(\mathbb{R}^n)$ with $\lambda_n(E) < \infty$ and $\varepsilon > 0$, $\exists\, g \in C_c^0(\mathbb{R}^n)$ s.t

$$\|\chi_E - g\|_p < \varepsilon$$

Step-2: $\exists\, g \in C_c(\mathbb{R}^n)$ s.t $g(K) = 1$ and $\text{supp}(g) \subset U$

where $K \subset E \subset U$

Exp: complete above theorm for $p = \infty$

Ans:

$S \subseteq L^\infty(\Omega)$ is trivial

$f \in L^\infty(\Omega)$

then from above $\exists\, \{\psi_n\}_{n=1}^{\infty} \subseteq S$

s.t $0 \leq |\psi_1| \leq \dots \leq |f|$

and $\psi_n(\omega) \to f(\omega)$ as $n \to \infty$ a.e — ①

we want to show $\|\psi_n\|_\infty \to \|f\|$ as $n \to \infty$

i.e

$$\|\psi_n - f\|_\infty \to 0 \text{ as } n \to \infty$$

or, $|\psi_n(\omega) - f(\omega)| \to 0$ as $n \to \infty$ a.e

but from above ①, we know $|\psi_n(\omega) - f(\omega)| \to 0$ as $n \to \infty$

$$\Rightarrow \|\psi_n - f\|_\infty \overset{a.e}{\to} 0 \text{ as } n \to \infty$$

30th March:

$f \in L^1(\Omega)$, and $f \geq 0$, then we can define

$$\nu(E) = \int_E f \, d\mu$$

from an exercise done, we have already seen that $\nu(E)$ is a finite measure on $(\Omega, \mathcal{H})$ ← dual

we want to show that $L^p(\Omega)^* \cong L^q(\Omega)$ $\quad 1/p + 1/q = 1$

Note: If $f \notin L^1(\Omega)$, then ν is a measure (not necessarily finite) but if f is s.t $f : \Omega \to \mathbb{R} / \bar{\mathbb{R}} / \mathbb{C} / \bar{\mathbb{C}}$

then ν fails to be a measure

now, let $f \in L^1(\Omega)$ and $f : \Omega \to \bar{\mathbb{R}}$ then

$$\nu^+(E) = \int_E f^+ \, d\mu$$

$$\nu^-(E) = \int_E f^- \, d\mu$$

now, let $\nu(E) = \nu^+(E) - \nu^-(E)$ or $\mu_f(E)$

if $f : \Omega \to \bar{\mathbb{R}}$ is measurable, then we need to assume that

$\nu^-(E) < \infty \quad \forall E \in \mathcal{H}$ ← just a sufficient, we can also have $E \in \mathcal{H}$ $\nu^+(E)$ and $\nu^-(E)$ both must be ∞

$\nu(\phi) = 0$

ν is countably additive

now, we don't have monotonicity

Defn: A function $\nu : \mathcal{H} \to \bar{\mathbb{R}}$ is called a signed measure if

1) $\nu(\phi) = 0$

2) ν is σ-additive

Defn: let $\nu : \mathcal{H} \to (-\infty, \infty]$, then a set $P \in \mathcal{H}$ is said to be a positive set if $\nu(E) \geq 0 \ \forall E \subseteq P$

similarly, a set $N \in \mathcal{H}$ is said to be negative set if $\nu(F) \leq 0 \ \forall F \subseteq N$

a set $\emptyset \in \mathcal{H}$ is said to be a null set if $\nu(A) = 0 \ \forall A \subseteq \emptyset$

2nd April:

Signed measure and Complex measure:

Defn: Suppose $(\Omega, \mathcal{H})$ is a measurable space, then signed measure $\nu : \mathcal{H} \to \bar{\mathbb{R}}$ is a function s.t

ν:

1) $\nu(\phi) = 0$
2) ν is σ-additive

Let $f : \Omega \to \bar{\mathbb{R}}$ be a measurable function. Then

$\nu(E) = \int_E f \, d\mu$ is a signed measure

provided $\int_E f^+ d\mu < \infty$ or $\int_E f^- d\mu < \infty$

Moreover if f is integrable, then $\mu(E) = \int_E f \, d\mu$ is a finite signed measure

Defn: Suppose $(\Omega, \mathcal{H})$ is a measurable space, then complex measure $\nu : \mathcal{H} \to \mathbb{C}$ is a function s.t

1) $\nu(\phi) = 0$
2) ν is σ-additive

Note: suppose $E \in \mathcal{H}$ with $\nu(E) = \infty$ and let $F \in \mathcal{H}$ s.t $E \cap F = \phi$

as $\nu(\underbrace{E \cup F}_{\in \mathcal{H}}) = \underbrace{\nu(E)}_{\infty} + \nu(F)$

$\Rightarrow \nu(F) > -\infty$ otherwise not defined

$\Rightarrow \nu(F) \in (-\infty, \infty]$

So if we have atleast one set with $\mu(E) = \pm\infty$ then, there does not exist any set with $\mu(F) = \mp\infty$

So, $\nu : \mathcal{H} \to (-\infty, \infty]$ wlog

Note: for a disjoint collection $\{E_i\}_{i=1}^{\infty} \subset \mathcal{H}$ we have

$\nu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \nu(E_i)$, as $\mu(\bigcup_{i=1}^{\infty} E_i)$ does not depend on rearrangement, the RHS is either absolutely convergent or one of the following is a real number

$\sum_{\nu(E_i) \geq 0} \nu(E_i)$ or $\sum_{\nu(E_i) < 0} \nu(E_i)$ $\left(\because \infty \text{ and } > -\infty \text{ or } < \infty \text{ and } -\infty\right)$

Lemma: Suppose ν is a signed measure on $(\Omega, \mathcal{H})$, let $E, F \in \mathcal{H}$ s.t $E \subset F$, then

1) if $|\nu(E)| < \infty$, then $\nu(F \setminus E) = \nu(F) - \nu(E)$
2) if $\nu(E) = \infty$, then $\nu(F) = \infty$
3) if $\nu(E) = -\infty$, then $\nu(F) = -\infty$

proof:

1) $\nu(F) = \nu(F \setminus E) + \nu(E)$, now as $|\nu(E)| < \infty$ we have

$$\nu(F\setminus E)=\nu(F)-\nu(E)$$

2) now $\nu(E)=+\infty$
then $\nu(F\setminus E)>-\infty$

$$\Rightarrow \underbrace{\nu(F\setminus E)}_{>-\infty}+\underbrace{\nu(E)}_{+\infty}=+\infty$$
$$\Rightarrow \nu(F)=+\infty$$

3) It follows similarly to 2

Lemma: Suppose $E\in\mathcal{H}$ and $\nu(E)=+\infty$, then $\forall\ F\in\mathcal{H}$, $\nu(F)>-\infty$

proof:
for case of $E\cap F=\phi$, we have already seen this is true

now, suppose $F\in\mathcal{H}$ s.t

$$E\cap F\neq\phi$$

then, $\left.\begin{array}{l}\nu(E\cap F)=+\infty\\ \nu(E\cap F)=-\infty\\ \nu(E\cap F)\in\mathbb{R}\end{array}\right\}$ are the three possibilities

case 1: $\nu(E\cap F)=\infty$
then as $E\cap F\subseteq F$
by previous result
$$\nu(F)=\infty$$

case 2: $\nu(E\cap F)=-\infty$
then $E\cap F\subseteq E$
then by previous lemma
$\nu(E)=-\infty$ $(\because \nu(E\cap F)=-\infty)$

But $\nu(E)=\infty$

case 3: if $\nu(E\cap F)\in\mathbb{R}$

$$\nu(E)=\nu(E\setminus F)+\nu(E\cap F)$$
$$\Rightarrow +\infty=\nu(E\setminus F)+\underbrace{\nu(E\cap F)}_{\in\mathbb{R}}$$
$$\Rightarrow \nu(E\setminus F)=+\infty$$





now, $\nu(E\cup F)=\underbrace{\nu(E\setminus F)}_{+\infty}+\nu(F)$

$$\Rightarrow \nu(F)>-\infty \text{ as } \nu(E\setminus F)=+\infty$$

so, from case 1,2,3 $\nu(F)>-\infty\ \forall\ F\in\mathcal{H}$

Defn: Suppose $\nu:\mathcal{H}\to(-\infty,\infty]$ is a signed measure on $(\Omega,\mathcal{H})$

a) a set $P\in\mathcal{H}$ is said to be positive if $\nu(E)\geq 0\ \forall E\in P$ and $E\in\mathcal{H}$

b) a set $N\in\mathcal{H}$ is said to be negative if $\nu(F)\leq 0\ \forall F\subseteq N$ and $F\in\mathcal{H}$

c) A set $O\in\mathcal{H}$ is said to be a null set if $\nu(E)=0\ \forall E\subseteq O$ and $E\in\mathcal{H}$

Note: positive sets is also ν-positive, same with ν-negative and ν-null

Theorem: (Hahn decomposition) If $\nu: \mathcal{A} \to (-\infty, \infty]$ is a signed measure on $(\Omega, \mathcal{A})$ then $\exists$ a ν-positive set $P \in \mathcal{A}$ s.t $N = \Omega \setminus P$ is a ν-negative set

Theorem: (Jordan-decomposition) Suppose $\nu: \mathcal{A} \to (-\infty, \infty]$ is a signed measure on $(\Omega, \mathcal{A})$ and (P, N) is a Hahn decomposition

$$\nu^+(E) = \nu(E \cap P) \text{ and}$$
$$\nu^-(E) = -\nu(E \cap N)$$
$$\forall E \in \mathcal{A}$$

then, ν^+ and ν^- are measures on $(\Omega, \mathcal{A})$ with $\nu = \nu^+ - \nu^-$
furthermore, ν^+, ν^- do not depend on the Hahn decomposition and will be unique

Defn: A signed measure ν is said to be

1) finite if ν^+, ν^- are finite

2) σ-finite if ν^+ and ν^- are σ-finite

Defn: $|\nu|(E) = \nu^+(E) + \nu^-(E)$ $\forall E \in \mathcal{A}$, $|\nu|$ is called the total variation of ν

Note: $|\nu(E)| \leq |\nu|(E)$ $\forall E \in \mathcal{A}$

Now, for complex measures, i.e on $(\Omega, \mathcal{A})$

$$\nu: \mathcal{A} \to \mathbb{C} \quad \text{s.t}$$
1) $\nu(\phi) = 0$
2) ν is σ-finite

Note: $\forall E \in \mathcal{A}$, $\mathrm{Re}(\nu(E))$ and $\mathrm{Im}(\nu(E))$ define signed measures on $(\Omega, \mathcal{A})$

Note: $\forall E \in \mathcal{A}$ let $\nu_1(E) = \mathrm{Re}(\nu(E))$
$\nu_2(E) = \mathrm{Im}(\nu(E))$
and $\nu = \nu_1 + i\nu_2$

then, $\forall E \in \mathcal{A}$, $|\nu(E)| = |\nu_1(E) + \nu_2(E)|$

$$\leq |\nu_1(E)| + |\nu_2(E)|$$
$$\leq |\nu_1|(E) + |\nu_2|(E)$$
$$\leq |\nu_1|(\Omega) + |\nu_2|(\Omega) < \infty$$

as $\nu: \mathcal{A} \to \mathbb{C}$ not $\overline{\mathbb{C}}$

Defn: The total variation of a complex measure $\nu: \mathcal{A} \to \mathbb{C}$ is defined by

$$|\nu|(E) = \sup \left\{ \sum_{j=1}^{\infty} |\nu(E_j)| \;\middle|\; E = \bigcup_{j=1}^{\infty} E_j, \text{ and } \{E_j\}_{j=1}^{\infty} \text{ is a disjoint collection in } \mathcal{A} \right\}$$

Note: $|\nu|(E)$ is s.t
$\forall c \in \mathbb{C}$ $|c\nu|(E) = |c|\,|\nu|(E)$
for ν_1, ν_2
$|\nu_1 + \nu_2|(E) \leq |\nu_1|(E) + |\nu_2|(E)$
So, set of ν's is also a normed vector space over $\mathbb{C}$

Theorem: $|\nu|$ is a finite measure on $(\Omega, \mathcal{A})$ and set of all complex measures on $(\Omega, \mathcal{A})$ is denoted by $M(\Omega)$, then $M(\Omega)$ is a vector space over $\mathbb{C}$ and

$$\|\nu\| = |\nu|(\Omega)$$

then, $(M(\Omega), \|\cdot\|)$ is a complete normed linear space

dual of $L^1 \to L^\infty$

dual of $L^\infty \to CM(\Omega), \|\cdot\|$

6th April:

Radon-Nikodym theorem and applications:

Defn: Suppose ν is a complex/signed measure $(\Omega, \mathcal{H})$, and μ is a measure on $(\Omega, \mathcal{H})$ ν is called absolutely continious w.r.t μ if $\nu(E)=0 \ \forall E \in \mathcal{H}$ with $\mu(E)=0$

then, $\nu \ll \mu$

Eg: Suppose $(\Omega, \mathcal{H}, \mu)$ is a measure space, fix some $f \in L^1(\Omega)$ and let

$$\mu_f(E) = \int_E f \, d\mu \Rightarrow |\mu_f(E)| \leq \|f\|_1 \mu(E)$$

then μ_f is complex measure on $(\Omega, \mathcal{H})$
and
$\mu_f(E) = 0$ for $\mu(E)=0 \Rightarrow \mu_f \ll \mu$

Eg: If ν is a complex measure on $(\Omega, \mathcal{H})$ then $\mathrm{Re}(\nu) \ll |\nu|$ and $\mathrm{Im}(\nu) \ll |\nu|$

Eg: any measure is absolutely cont wrt counting measure

Theorm: (Radon-Nikodym) Suppose μ is a σ-finite measure on $(\Omega, \mathcal{H})$ and also suppose ν is a complex measure on $(\Omega, \mathcal{H})$ s.t $\nu \ll \mu$, then $\exists f \in L^1(\Omega, \mathcal{H}, \mu)$ s.t $\nu(E) = \int_E f \, d\mu \ \forall E \in \mathcal{H}$, $f := \frac{d\nu}{d\mu}$ is called the Radon-Nikodym derivative of ν wrt μ

Note: Above theorm is false if μ is not finite, as $\mathcal{H} = \mathcal{B}(\mathbb{R})$, $C = \mu$, $\nu = \lambda$ then $\lambda \ll C$

$C(\{x\}) = 1$ and $\lambda(\{x\}) = 0$

$0 = \lambda(\{x\}) = \int_{\{x\}} f \, dC = f(x)$ let $f: \mathbb{R} \to \bar{\mathbb{R}}$ be $\frac{d\lambda}{dC}$ (C: counting measure)

$\Rightarrow f \equiv 0$ i.e $f(x)=0 \ \forall x \in \mathbb{R}$

but then $\lambda(E) = \int_E 0 \, dC = 0$ not true say for $E=[0,1]$

Note: In case of complex measure everything is finite as $\nu: \mathcal{H} \to \mathbb{C}$ and not $\bar{\mathbb{C}}$

Defn: $(\Omega, \mathcal{H}, \mu)$, let $p \in [1, \infty]$, a continious/bdd linear functional on $L^p(\Omega)$ is a linear map

$$\varphi : L^p(\Omega) \to \mathbb{C} \text{ s.t } \sup \left\{ \frac{|\varphi(f)|}{\|f\|_p} \;\middle|\; f \in L^p(\Omega) \text{ and } 0 < \|f\|_p < 1 \right\} < \infty$$

equivalent to $M = \sup \{ |\varphi(f)| \mid f \in L^p(\Omega) \text{ and } \|f\|_p = 1 \} < \infty$

let $M = \|\varphi\|_{L^p(\Omega)^*}$

Note: continious and bdd is equivalent for linear functionals on $L^p(\Omega)$

Defn: $L^p(\Omega)^* = \{ \varphi : L^p(\Omega) \to \mathbb{C} \mid \varphi \text{ is cont/bdd} \}$, then $L^p(\Omega)^*$ is called the dual of $L^p(\Omega)$

Exe: show that $(L^p(\Omega)^*, \| \|_{L^p(\Omega)^*})$ is a normed linear space → done down

Note: fix $p \in [1, \infty)$, now suppose $g \in L^q(\Omega)$ where $\frac{1}{p} + \frac{1}{q} = 1$, now let

$\varphi_g : L^p(\Omega) \to \mathbb{C}$ by $\varphi_g(f) = \int_\Omega fg\,d\mu$

(linear functional)

if $p=1, q=\infty$ then $g \in L^\infty(\Omega) \Rightarrow |\varphi_g(f)|$

$\leq \|g\|_\infty \int_\Omega |f|\,d\mu$

$< \infty$ as $f \in L^1(\Omega)$

$p \neq 1, q \neq \infty$: by Holder's inequality

$$|\varphi_g(f)| \leq \|f\|_p \|g\|_q$$

$$\Rightarrow \sup\left\{ |\varphi_g(f)| \;\middle|\; f \in L^p(\Omega), \|f\|_p = 1 \right\}$$

$$= \|\varphi_g\|_{L^p(\Omega)^*} \leq \|g\|_q$$

$$\Rightarrow \varphi_g \in L^p(\Omega)^*$$

Defn: $\langle g, f \rangle = \varphi_g(f) = \int_\Omega fg\,d\mu$ is called a dual pairing

Let $f(x) = |g(x)|^{q-1} \left(\frac{g(x)}{|g(x)|} \right)^{-1}$

$\Rightarrow |f(x)|^p = |g(x)|^{p(q-1)} = |g(x)|^q$ as $p+q = pq$

$\Rightarrow \|f\|_p^p = \|g\|_q^q$

$\Rightarrow \|f\|_p = \|g\|_q^{q/p}$, then

$$\varphi_g(f) = \int_\Omega fg\,d\mu$$

$$= \int_\Omega |g(x)|^{q-1} \left(\frac{g(x)}{|g(x)|} \right)^{-1} g(x)\,d\mu$$

$$= \int_\Omega |g(x)|^q\,d\mu = \|g\|_q^q$$

$$= \|g\|_q^{q-1} \|g\|_q$$

now $\|f\|_p = \|g\|_q^{q/p}$

$\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{q}{p} + 1 = q$

$\Rightarrow \frac{q}{p} = q - 1$

$\Rightarrow \|f\|_p = \|g\|_q^{(q-1)}$

So, $\varphi_g(f) = \|f\|_p \|g\|_q$

$\Rightarrow \|\varphi_g\|_{L^p(\Omega)^*} = \|g\|_q$

ie $L^q(\Omega) \ni g \to \varphi_g \in L^p(\Omega)^*$ is a linear isometry ($\because$ it preserves the norm)

Theorem: $p \in [1,\infty)$, suppose μ is a σ-finite measure on $(\Omega, \mathcal{A})$, for $g \in L^q(\Omega)$ let $\varphi_g \in L^p(\Omega)^*$ by $\varphi_g(f) = \int f g \, d\mu$, then

$g \longrightarrow \varphi_g$ is bijective linear isometry

If we consider $(1,\infty)$ then dual of L^p is L^q and of L^q is L^p
bidual is isomorphic to original space

if $p=1$ then dual is L^∞, dual of L^∞ is set of all measures on Ω (Radon measures)

Exe: show that $(L^p(\Omega)^*, \| \|_{L^p(\Omega)^*})$ is a normed linear space

Ans: $f, g \in L^p(\Omega)^*$

$$\| f+g \|_{L^p(\Omega)^*} \leq \| f \|_{L^p(\Omega)^*} + \| g \|_{L^p(\Omega)^*}$$

$$\Rightarrow f+g \in L^p(\Omega)^*$$

$$\| fg \|_{L^p(\Omega)^*} \leq \| f \|_{L^p(\Omega)^*} \| g \|_{L^p(\Omega)^*}$$

$$\Rightarrow fg \in L^p(\Omega)^*$$

9th April:

Theorem: Suppose $(\Omega, \mathcal{H}, \mu)$ is a measure space, fix $p \in (1,\infty)$, there is a bijection b/w $L^p(\Omega)^*$ and $L^q(\Omega)$ where $\frac{1}{p}+\frac{1}{q}=1$, moreover if μ is σ-finite, then $(L^1(\Omega))^*$ and $L^\infty(\Omega)$ are in bijection

the bijection is given by the following way

$\forall\, \varphi \in L^p(\Omega)^*$, then $\exists\, g \in L^q(\Omega)$ s.t $\varphi(f) = \int_\Omega fg\, d\mu$

proof: **case-1:** $\mu(\Omega) < \infty$, let $p \in (1,\infty)$

Suppose $\varphi \in L^p(\Omega)^*$ let

$$\nu(E) = \varphi(\chi_E) \quad \forall E \in \mathcal{H}$$

then $\nu(\phi) = \varphi(0) = 0$

suppose $\{E_i\}_{i=1}^{\infty}$ is a disjoint seq in $\mathcal{H}$

then $\sum_{k=1}^{\infty} \chi_{E_k} \longrightarrow \chi_{\bigcup_{k=1}^{\infty} E_k}$ in L^p-norm as $n \to \infty$ (∵ below exercise)

then $\nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \varphi\left(\chi_{\bigcup_{k=1}^{\infty} E_k}\right) = \varphi\left(\lim_{n\to\infty} \sum_{k=1}^{n} \chi_{E_k}\right)$

$$= \lim_{n\to\infty} \varphi\left(\sum_{k=1}^{n} \chi_{E_k}\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \varphi(\chi_{E_k})$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \nu(E_k)$$

$$= \sum_{k=1}^{\infty} \nu(E_k)$$

so, ν is a complex measure on $(\Omega, \mathcal{H})$

for $E \in \mathcal{H}$ with $\mu(E) = 0$ we have $\chi_E = 0$ in $L^p(\Omega)$

$\Rightarrow \nu(E) = \varphi(\chi_E) = \varphi(0) = 0$

$\Rightarrow \nu \ll \mu$

then by radon-nikodym theorm, $\exists\, g \in L^1(\Omega)$ s.t

$$\nu(E) = \int \chi_E g\, d\mu$$

so, $\varphi(f) = \int_\Omega fg\, d\mu$ $\forall$ basic functions $f : \Omega \to \mathbb{C}$

now, every bounded measurable functions can be approximated uniformly by basic functions

so, $\varphi(f) = \int_\Omega fg\, d\mu \quad \forall f \in L^\infty(\Omega)$

↑ f is bounded measurable

for $k \in \mathbb{Z}^+$ let $E_k = \{\omega \in \Omega \mid 0 < |f(\omega)| \leq k\}$

$$f_K(\omega) = \begin{cases} \overline{g(\omega)}\,|g(\omega)|^{q-2} & \text{if } \omega \in E_K \\ 0 & \text{otherwise} \end{cases} \qquad \{f_K\}_{K=1}^{\infty} \subseteq L^p(\Omega)$$

$f_K \in L^p(\Omega)$ from construction

consider $\int_\Omega |g|^q \chi_{E_K}\, d\mu = \varphi(f_K)$ ($\because$ trivial)

$\leq \|\varphi\|_{L^p(\Omega)^*} \|f_K\|_p$ ($\because$ defn of $\|\varphi\|_{L^p(\Omega)^*}$)

$$\|f_K\|_p = \left(\int_{E_K} |g|^{(q-1)p}\, d\mu\right)^{\frac{1}{p}}$$

$$= \|\varphi\|_{L^p(\Omega)^*} \left(\int_{E_K} |g|^q\, d\mu\right)^{\frac{1}{p}}$$

$\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow (q-1) = \frac{q}{p}$

$$= \|\varphi\|_{L^p(\Omega)^*} \left(\int_\Omega |g|^q \chi_{E_K}\, d\mu\right)^{\frac{1}{p}}$$

$$\Rightarrow \left(\int_\Omega |g|^q \chi_{E_K}\, d\mu\right)^{\frac{1}{q}} \leq \|\varphi\|_{L^p(\Omega)^*}$$

as $K \to \infty$

$$\Rightarrow \|g\|_q \leq \|\varphi\|_{L^p(\Omega)^*}$$

$\Rightarrow g \in L^q(\Omega)$ — ①

let $f \in L^p(\Omega)$, $f_n(x) = \begin{cases} f(x) ; & |f(x)| \leq n \\ n \cdot \frac{f(x)}{|f(x)|} ; & |f(x)| > n \end{cases}$ ← this is to show $L^p(\Omega)$ is dense in $L^\infty(\Omega)$

$\{f_n\}_{n=1}^{\infty} \to L^\infty(\Omega)$

now, consider $\|f - f_n\|_p^p = \int_\Omega |f - f_n|^p\, d\mu$

$$= \int_{\{x \in \Omega \,|\, |f(x)| > n\}} |f - f_n|^p\, d\mu$$

$$(f - f_n)(x) = f(x) - f_n(x) = \frac{f(x)}{|f(x)|}\left(|f(x)| - n\right)$$

$\Rightarrow |f - f_n|(x) = |f(x) - n| = |f(x)| - n \quad \forall x \in \{\omega \in \Omega \,|\, |f(\omega)| > n\}$

$\Rightarrow \|f - f_n\|_p^p \leq \int_{\{\,\}} |f|^p d\mu$ as $n \to \infty$, $\|f - f_n\|_p \to 0$

$\Rightarrow f_n \to f$ in $L^p(\Omega)$

so, $\varphi(f) = \int_\Omega fg\, d\mu \quad \forall f \in L^p(\Omega)$ ($\because$ as L^∞ is dense in L^p so, we replace f in L^∞ by f in L^p)

and $g \in L^q(\Omega)$ ($\because$ seen above)

Case 2: $\mu(\Omega) < \infty$, $p = 1$, then this follows from tutorial

Case 3: μ is σ-finite, then $\exists$ an inc seq $\{E_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ s.t

$0 < \mu(E_n) < \infty \quad \forall n \in \mathbb{N}$

and $\Omega = \bigcup_{n=1}^{\infty} E_n$

for $m > n$, $E_n \subseteq E_m \subseteq \Omega$

$$L^r(E_n) \subseteq L^r(E_m) \subseteq L^r(\Omega) \quad \forall r \in [1, \infty)$$

↑ by setting 0 for $E_m \setminus E_n$

as, $\forall f \in L^r(E_n)$ then $\tilde{f} \in L^r(E_m)$ by

$$\tilde{f}(x) = \begin{cases} f(x) & ; x \in E_n \\ 0 & ; x \in E_m \setminus E_n \end{cases}$$

now, $\varphi \in L^p(\Omega)^*$, then $\varphi_k = \varphi|_{L^p(E_k)} \quad \forall k \in \mathbb{N}$

then, $\exists\, g_k \in L^q(E_k)$ s.t $\varphi_k(f) = \int_{E_k} f g_k \, d\mu \quad \forall f \in L^p(E_k)$ ($\because$ case 1, 2)

$$\varphi_m(f) = \int_{E_m} f g_m \, d\mu \quad \forall f \in L^p(E_m)$$

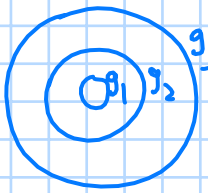
$$\varphi_n(f) = \int_{E_n} f g_n \, d\mu \quad \forall f \in L^p(E_n)$$

$$\left(\varphi_m|_{L^p(E_n)} - \varphi_n\right)(f) = \int_{E_n} f(g_m - g_n)\, d\mu \,, \quad \forall f \in L^p(E_n)$$

$= 0$ as f was trivial solution

so, $\int_{E_k} f(g_m - g_n)\, d\mu = 0 \quad \forall f \in L^p(E_n)$

$\Rightarrow g_m = g_n$ a.e on E_n



let $g : \Omega \to \mathbb{C}$ by $g|_{E_n} = g_n$ well defined

by monotone convergence theorem,

$$\|g\|_q = \lim_{n \to \infty} \|g_n\|_q \leq \|\varphi\| \Rightarrow g \in L^q(\Omega)$$

then by dominated convergence theorem, ← as $f\chi_{E_n} \to f$ in $L^p(\Omega)$

$\{f\chi_{E_n}\}_{n=1}^{\infty}$ $f\chi_{E_n} \to f$ in $L^p(\Omega)$ $\lim \int_{\Omega} f\chi_{E_n} g \, d\mu = \int_{\Omega} fg \, d\mu$

$$\varphi(f) = \lim_{n \to \infty} \varphi(f\chi_{E_n}) = \lim_{n \to \infty} \int_{E_n} fg \, d\mu = \int_{\Omega} fg \, d\mu$$

case 4: μ is a positive measure

fix $\varphi \in L^p(\Omega)^*$ $M = \sup\{ \|g_E\|_q \mid E \in \mathcal{A} \text{ is a } \sigma\text{-finite set} \}$

then $M \leq \|\varphi\|$

Exe: show that, $\{E_i\}_{i=1}^{\infty}$ is a disjoint seq in $\mathcal{A}$ → done down

then $\sum_{k=1}^{\infty} \chi_{E_k} \longrightarrow \chi_{\bigcup_{k=1}^{\infty} E_k}$ in L^p-norm as $n \to \infty$

Exe: let $\{f_n\}_{n=1}^{\infty}$ be a seq of basic fution s.t $f_n \to f$ in $L^p(\Omega)$ as $n \to \infty$,

prove or disprove that $\varphi(f_n) \to \varphi(f)$ as $n \to \infty$ → done down

Exe: show that $\{E_i\}_{i=1}^{\infty}$ is a disjoint seq in $\mathcal{H}$

then $\sum_{K=1}^{\infty} \chi_{E_K} \longrightarrow \chi_{\bigcup_{K=1}^{\infty} E_K}$ in L^p-norm as $n \to \infty$

Ans: $\{E_i\}_{i=1}^{\infty}$ is disjoint sequence in $\mathcal{H}$

now, let $f_n(\omega) = \sum_{K=1}^{n} \chi_{E_K}(\omega)$

$$f(\omega) = \chi_{\bigcup_{K=1}^{\infty} E_K}(\omega)$$

$$\|f_n - f\|_p^p = \int_{\Omega} \left| \sum_{K=1}^{n} \chi_{E_K} - \chi_{\bigcup_{K=1}^{\infty} E_K} \right|^p d\mu$$

$$= \int_{E_1} |1-1|^p d\mu + \int_{E_2} |1-1|^p d\mu + \int_{E_3} |1-1|^p d\mu + \cdots + \int_{E_n} |1-1|^p d\mu + \int_{\Omega} \left| \chi_{\bigcup_{K=n+1}^{\infty} E_K} \right|^p d\mu$$

$$\|f_n - f\|_p^p = \int_{\Omega} \left| \chi_{\bigcup_{K=n+1}^{\infty} E_K} \right|^p d\mu$$

$$\lim_{n \to \infty} \|f_n - f\|_p^p = 0 \quad \text{as } K \to \infty$$

$\Rightarrow f_n \to f$ in L^p norm as $n \to \infty$

Exe: Let $\{f_n\}_{n=1}^{\infty}$ be a seq of basic fution s.t $f_n \to f$ in $L^p(\Omega)$ as $n \to \infty$,

prove or disprove that $\varphi(f_n) \to \varphi(f)$ as $n \to \infty$

Ans:

$f_n \to f$ in $L^p(\Omega)$ and they are basic, as $n \to \infty$

let χ_E be a fution, then from above theorm

$\exists\, g \in L^q(\Omega)$ s.t $\varphi(f) = \int_{\Omega} f g \, d\mu$

moreover, $\varphi(\chi_E) = \int_{\Omega} \chi_E g \, d\mu = \int_E g \, d\mu$

now $\varphi(f_n) = \int_{\Omega} f_n g \, d\mu$

$$|\varphi(f_n) - \varphi(f)| = \left| \int_{\Omega} (f_n - f) g \, d\mu \right|$$

$$\leq \|f_n - f\|_p \, \|g\|_q$$

now as $\|g\|_q < \infty$ and $f_n \to f$ in $L^p(\Omega)$

$\Rightarrow \lim_{n \to \infty} |\varphi(f_n) - \varphi(f)| = 0$

$\Rightarrow \varphi(f_n) \longrightarrow \varphi(f)$